Quantum Macrostatistical Theory of Nonequilibrium Steady States

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Abstract. We provide a general macrostatistical formulation of nonequilibrium steady states of reservoir driven quantum systems. This formulation is centred on the large scale properties of the locally conserved hydrodynamical observables, and our basic physical assumptions comprise (a) a chaoticity hypothesis for the nonconserved currents carried by these observables, (b) an extension of Onsager's regression hypothesis to fluctuations about nonequilibrium states, and (c) a certain mesoscopic local equilibrium hypothesis. On this basis we obtain a picture wherein the fluctuations of the hydrodynamical variables about a nonequilibrium steady state execute a Gaussian Markov process of a generalized Onsager-Machlup type, which is completely determined by the position dependent transport coefficients and the equilibrium entropy function of the system. This picture reveals that the transport coefficients satisfy a generalized form of the Onsager reciprocity relations in the nonequilibrium situation and that the spatial correlations of the hydrodynamical observables are generically of long range. This last result constitutes a model-independent quantum mechanical generalization of that obtained for special classical stochastic systems and marks a striking difference between the steady nonequilibrium and equilibrium states, since it is only at critical points that the latter carry long range correlations.

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1. Introduction

The statistical thermodynamics of nonequilibrium steady states or, more generally, dynamically stable ones, of reservoir driven macroscopic systems[b] is a key area of the natural sciences, with ramifications for condensed matter physics [1-4], chemistry [5] and biology [6]. At the phenomenological and heuristic levels, there is an abundant literature on this subject. At the level of mathematical physics, however, the subject is still at an exploratory stage. In the classical regime, two types of rigorous approaches have been made to it. The first is centred on the hypotheses that the macroscopic properties of complex systems are yielded by the model of classical Anosov dynamical systems [7,8]. This hypothesis is designed to capture the chaoticity that underlies macroscopic irreversibility, and it has been shown to lead to nonequilibrium generalizations both of the Onsager reciprocity relations [8] and of the fluctuation-dissipation theorem [7]. A second approach is centred on microscopic treatments of stochastic (non-Hamiltonian) dynamical models [9-11], which are also designed to capture the chaoticity underlying macroscopic irreversibility. The treatment of these models has led to some interesting developments, and Ref. [11] has provided a dynamically based picture of the hydrodynamical fluctuations about their nonequilibrium steady states. Moreover, in the case of a certain particular model, namely the symmetric exclusion process, it has been shown that the nonequilibrium steady state has long range density correlations [9-11] and that the probability distribution of its large scale density field is determined by an explicitly specified and highly nontrivial nonequilibrium generalization of its free energy [10, 11]. In the quantum regime, a natural dynamically based definition of nonequilibrium steady states of reservoir driven systems has been formulated [12, 13] at the microscopic level.

In the present article we set out a different approach to the subject, which is *quantum macrostatistical* in that is is centred on the hydrodynamical observables of reservoir driven quantum systems. This approach, which was briefly sketched in Ref. [14], parallels the one we have previously made to the nonequilibrium thermodynamics of conservative quantum systems [15, 16], where it yielded an extension of the Onsager reciprocity relations to a nonlinear regime. In general, the quantum macrostatistics is designed, like Onsager's [17] irreversible thermodynamics and Landau's fluctuating hydrodynamics [18], to form a bridge between the microscopic and macroscopic pictures of matter, rather than a deduction of the latter from the former. Indeed, accepting Boltzmann's hypothesis of molecular chaos [19], we take the view that such a derivation is not even feasible for realistically interacting systems, since this chaos renders the microscopic equations of motion intractable over periods substantially longer than the intervals between successive collisions[c]. Thus, the microscopic equations of motion must necessarily be supplemented by further assumptions in order to interconnect the quantum and phenomenological properties

[b] A very simple example of such a state is the stationary one of a solid rod, whose ends are coupled to thermostats of different temperatures.

[c] This view is supported by the fact that the rigorous derivations of Boltzmann equations from the Hamiltonian dynamics of both classical [20] and quantum [21] systems are applicable only over microscopic times of the order of the interval between successive collisions of a particle. For longer times, the chaos bars the way to further analysis of the

of matter. In fact, the key physical assumptions of our macrostatistical project concern only very general, model-independent properties of many-particle systems. Specifically, they comprise

(A) an extension of Onsager's regression hypothesis [17], to the effect that the hydrodynamical fluctuations about nonequilibrium steady states are governed by the same dynamical laws as the 'small' perturbations of the hydrodynamical variables about their steady values;

(B) a certain mesoscopic local equilibrium hypothesis; and

(C) a chaoticity hypothesis for the nonconserved currents carried by the locally conserved hydrodynamical observables.

These assumptions may be regarded as the 'axioms' of our theory. The physical considerations that underlie them will be discussed, along with their formulation, in Sections 4.1, 4.1 and 4.4. In fact, the hypothesis (C), like Boltzmann's *Stosszahlansatz* and its subsequent developments [7-11], exploits the consequence of the very chaos that obstructs the analytical dynamics of realistically interacting many-particle systems.

The principal results that we obtain by supplementing the Schroedinger dynamics of many-particle systems by the 'axioms' (A)-(C), together with certain technical assumptions, are the following ones (I)-(III), which we claim to be new, at least on the level of a rigorous, general, model-independent quantum theory of nonequilibrium steady states.

(I) The spatial correlations of the hydrodynamical observables are generically of long range. This comprises a quantum mechanical generalization of that obtained from both rigorous microscopic treatments of certain classical stochastic models [9-11] and from heuristic treatments [23, 24] of Landau's fluctuating hydrodynamics. Most importantly, it marks a qualitative difference between equilibrium and nonequilibrium steady states, since the hydrodynamical correlations in the former states are generically of short range, except at critical points.

(II) The transport coefficients satisfy a generalized, position-dependent version of the Onsager reciprocity relations. Thus, this result extends Onsager's irreversible thermodynamics from the neighbourhood of equilibrium to that of nonequilibrium steady states.

(III) The hydrodynamical fluctuations execute a classical Gaussian Markov process, of a generalized Onsager-Machlup (OM) type [22]. Thus this result extends the OM theory from the regime of fluctuations about thermal equilibrium to that of fluctuations about nonequilibrium steady states. A similar result was obtained for certain classical stochastic models in Ref. [11].

Let us now briefly describe the macrostatistical strategy we employ to obtain these results. We take our model to be an N-particle quantum system, Σ, that is confined to a bounded open connected region, Ω_N, of a d-dimensional Euclidean space, X, and coupled at its boundary, $\partial\Omega_N$, to an array, $\mathcal{R}$, of quantum mechanical reservoirs. Σ is thus an open system, while the composite $(\Sigma + \mathcal{R})$ is a conservative one. Since we shall have occasion

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microscopic equations of motion.

to pass to thermodynamic and hydrodynamic limits where its particle number tends to infinity, we take N to be a variable parameter of the system. We assume that its particle number density $\nu := N/\mathrm{Vol}(\Omega_N)$ is N-independent and that Ω_N is the dilation by a factor L_N of a fixed, N- independent region Ω of unit volume. Thus $\Omega_N = L_N\Omega := \{L_N x | x \in \Omega\}$ and

$$L_N = (N/\nu)^{1/d}. \tag{1.1}$$

For the hydrodynamic description of Σ, we take L_N to be the unit of length. Thus, Ω is the region occupied by the system in the hydrodynamical picture.

We assume that, in that picture, Σ evolves according to a phenomenological law governing the evolution of a set of locally conserved classical fields $q_t(x) = \big(q_{1,t}(x), \ldots, q_{m,t}(x)\big)$, which correspond to the densities at position x and time t of the extensive thermodynamic variables[d] of the system. We denote the associated currents of $q_t(x)$ by $j_t(x) = \big(j_{1,t}(x), \ldots, j_{m,t}(x)\big)$. Thus, q_t satisfies the local conservation law

$$\frac{\partial q_t}{\partial t} + \nabla . j_t(x) = 0. \tag{1.2}$$

We assume that its phenomenological dynamics is governed by a constitutive equation of the form

$$j_t(x) = \mathcal{J}(q_t; x), \tag{1.3}$$

where $\mathcal{J}$ is a functional of the field q_t and the position x. Thus, by Eqs. (1.2) and (1.3), q_t evolves according to an autonomous law

$$\frac{\partial q_t(x)}{\partial t} = \mathcal{F}(q_t; x) := -\nabla . \mathcal{J}(q_t; x), \tag{1.4}$$

subject to boundary conditions determined by the reservoirs. We assume that this phenomenological law is invariant under scale transformations $x \to \lambda x,\ t \to \lambda^k t$ for some constant k. A simple example for which this assumption is valid, with $k = 2$, is that of nonlinear diffusions, where $\mathcal{J}$ takes the form

$$\mathcal{J}(q_t; x) = -\tilde{K}\big(q_t(x)\big)\nabla q_t(x), \tag{1.5}$$

$\tilde{K}$ being an m-by-m matrix $[\tilde{K}_{kl}]$, which acts by standard matrix multiplication on ∇q_t. In this case, the phenomenological equation (1.4) takes the form

$$\frac{\partial q_t}{\partial t} = \nabla . \big(\tilde{K}(q_t)\nabla q_t\big). \tag{1.6}$$

We shall base some of our explicit calculations on this case and, in particular, we shall henceforth assume that the scaling exponent k is equal to 2. A simple consequence of this

[d] We provide a characterization of these variables in Section 2.2 along lines previously formulated in Ref. [15].

assumption is that, since L_N is the unit of length for the hydrodynamical picture, L_N^2 is the unit of time for this picture.

We assume that, in general, the dynamics described by Eq. (1.4) is dissipative, in that the m-component field $q_t(x)$ relaxes eventually to a unique time-independent form $q(x)$, which thus corresponds to a steady hydrodynamical state. By Eq. (1.3), the corresponding steady m-component current, $j(x)$, is then $\mathcal{J}(q;x)$.

By Eq. (1.4), the linearised equation of motion for 'small' perturbations, $\delta q_t(x)$, of $q(x)$ is simply

$$\frac{\partial}{\partial t}\delta q_t(x) = \mathcal{L}\delta q_t(x) := \frac{\partial}{\partial \lambda}\mathcal{F}(q+\lambda\delta q_t;x)_{|\lambda=0}, \tag{1.7}$$

while, by Eq. (1.3), the corresponding increment in the m- component current $j(x)$ is

$$\delta j_t(x) = \mathcal{K}\delta q_t(x) := \frac{\partial}{\partial \lambda}\mathcal{J}(q+\lambda\delta q_t;x)_{|\lambda=0}. \tag{1.8}$$

We note that, by Eqs. (1.4), (1.7) and (1.8),

$$\mathcal{L} = -\nabla.\mathcal{K}. \tag{1.9}$$

Further, in the case of nonlinear diffusions, it follows from the identification of the r.h.s.'s of Eqs. (1.4) and (1.6) that Eq. (1.7) yields the following formal equation for $\mathcal{L}$.

$$[\mathcal{L}\chi](x) = \nabla.\Big(\tilde{K}\big(q(x)\big)\nabla\chi(x) + \big[\tilde{K}'\big(q(x)\big)\chi(x)\big]\nabla q(x)\Big), \tag{1.10}$$

where χ is a single column matrix function of position and $\tilde{K}'(q)$ is the derivative of $\tilde{K}(q)$, i.e. its gradient with respect to q: thus $[\tilde{K}'(q)\chi(q)]_{kl} = \sum_{r=1}^m[\partial\tilde{K}_{kl}(q)/\partial q_r]\chi_r(q)$.

In order to relate the phenomenological dynamics given by Eqs. (1.4) and (1.7) to the underlying microscopic quantum mechanics of Σ, we assume that $q_t(x)$ is the expectation value of a set of locally conserved quantum fields $\hat{q}_t(x) = \big(\hat{q}_{1,t}(x), . ., \hat{q}_{m,t}(x)\big)$ as rescaled for the hydrodynamical picture and in a limit in which N, and hence L_N, becomes infinite. Correspondingly, we formulate the fluctuations $\xi_t(x)$ of this m-component quantum field $q_t(x)$ about its mean on the same macroscopic scale and with a standard normalization, subject to the above-described assumptions (A)-(C).

On this basis, we establish that ξ_t executes a Gaussian Markov process represented by a generalized Langevin equation of the form

$$\frac{\partial}{\partial t}\xi_t(x) = \mathcal{L}\xi_t(x) + b_t(x), \tag{1.11}$$

where $b_t(x)$ is a white noise whose autocorrelation function is of zero range with respect to position as well as time. Thus, ξ_t executes a generalized Onsager-Machlup process. We employ this result to infer that the spatial correlations of the fluctuation field ξ in nonequilibrium steady states are generically of long range. In this way we derive the above results (I)-(III) from our basic macrostatistical assumptions.

We present our treatment as follows. In Section 2 we formulate the quantum statistical thermodynamical model of the composite system $(\Sigma+\mathcal{R})$ at both microscopic and macroscopic levels. This formulation provides general specifications of the nonequilibrium steady states of the model and also of the locally conserved quantum fields $\hat{q}_t$ and associated currents $\hat{j}_t$ pertinent to its hydrodynamic description. Here, in accordance with the general requirements of quantum field theory [25], we assume that these are distribution-valued operators. In Section 3 we relate the classical hydrodynamical variables, q_t and j_t, and their fluctuations, ξ_t and η_t, about a nonequilibrium steady state to these quantum fields and currents; and we obtain sufficient conditions for the fluctuations ξ_t to execute a *classical* stochastic process. In Section 4 we formulate our regression and local equilibrium hypotheses for this process and note that these, together with the assumption of microscopic reversibility for the composite $(\Sigma+\mathcal{R})$, yields a canonical extension of Onsager's reciprocity relations to the nonlinear hydrodynamical regime. In Section 5 we extend our local equilibrium hypothesis to the fluctuating currents, η_t, and formulate our chaoticity hypothesis for these currents. We then establish that the assumptions of the regression hypothesis, local equilibrium and chaoticity imply the field ξ_t executes a generalized Onsager-Machlup process represented formally by Eq. (1.11). In Section 6 we obtain an explicit formula for the two-point function for this process in terms of the equilibrium entropy density function and the transport coefficients of the system, and we infer therefrom that the static correlations of the hydrodynamical fluctuation field ξ are generically of non-zero range on the macroscopic scale and hence of long (infinite!) range on the microscopic one. We conclude in Section 7 with some general observations about the results of this article and of their possible generalizations to less restrictive conditions than those assumed here. We leave the proofs of some technical Propositions to four Appendices.

2. The Quantum Model.

We take our model to be the open quantum system, Σ, briefly described in Section 1. Thus, Σ is a system of N particles, which occupies a bounded open connected region, Ω_N, of a d-dimensional Euclidean space X and is coupled at its surface, $\partial\Omega_N$, to an array, $\mathcal{R}$, of reservoirs. Here Ω_N is the dilation by a factor L_N of a region, Ω, of unit volume and L_N is given by Eq. (1), which represents the N-independence of the particle density of Σ. We assume that the composite quantum system $\Sigma^{(c)} := (\Sigma+\mathcal{R})$ is conservative and that all its interactions are invariant under spatial translations and rotations.

2.1. The Microscopic Picture. We formulate this picture in standard operator algebraic terms, denoting the $C^{\star}$-algebras of bounded observables of Σ and $\Sigma^{(c)}$ by $\mathcal{A}$ and $\mathcal{B}$, respectively. We assume that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and that it is isomorphic to the $W^{\star}$-algebra of bounded operators in a separable Hilbert space $\mathcal{H}$, which comprises the square integrable functions $f(x_1,.\,,x_N;s_1,.\,.,s_N)$ (appropriately symmetrized or antisymmetrized) of the positions $\{x_j\}$ and the spins $\{s_j\ (=\pm 1)\}$ of its particles. The unbounded observables of Σ are represented by the unbounded self-adjoint operators affiliated to $\mathcal{A}$, i.e. by those whose spectral projectors belong to this algebra. The states of this system are represented by the density matrices in $\mathcal{H}$, and the expectation value of an observable, A, of Σ for the state ρ is $\mathrm{Tr}(\rho A)$. In general we denote this expectation value by $\rho(A)\equiv\langle\rho;A\rangle$,

and we employ the corresponding notation for $\Sigma^{(c)}$.

The Wigner time reversal operator, which serves to reverse the velocities and spins of the particles of Σ, is defined to be the antilinear transformation of $\mathcal{H}$ given by the formula

$$(Tf)(x_1,.\,,x_N;s_1,.\,.,s_N) = \overline{f}(x_1,.\,,x_N;-s_1,.\,.,-s_N)\ \forall f\in\mathcal{H}, \tag{2.1}$$

where the bar denotes complex conjugation. Thus, T implements an antiautomorphism $\tau_{\mathcal{A}}$ of $\mathcal{A}$, defined by the formula

$$\tau_{\mathcal{A}}A = TA^{\star}T\ \forall\ A\in\mathcal{A}. \tag{2.2}$$

We assume that the dynamics of the composite system $\Sigma^{(c)}$ is given by a one-parameter group, $\{\alpha_t|t\in\mathbf{R}\} := \alpha(\mathbf{R})$, of automorphisms of $\mathcal{B}$. Further, we assume that this dynamics is reversible, i.e. that $\mathcal{B}$ is equipped with an antiautomorphism τ, which reduces to $\tau_{\mathcal{A}}$ on $\mathcal{A}$ and implements time reversals according to the prescription

$$\tau\alpha_t\tau = \alpha_{-t}. \tag{2.3}$$

The evolution of the observables of Σ is given by the isomorphisms of $\mathcal{A}$ into $\mathcal{B}$ obtained by the restriction of $\alpha(\mathbf{R})$ to the former algebra.

2.2. Thermodynamic Variables and Potentials. In order to formulate the thermodynamic observables and potentials of Σ we pass, for the moment, to the situation where it is decoupled from the reservoirs $\mathcal{R}$ and thus becomes a conservative system, whose dynamics is given by a one-parameter group, $\{\alpha_t^{(0)}|t\in\mathbf{R}\}$, of automorphisms of $\mathcal{A}$. In this situation, its canonical equilibrium state, ρ, at inverse temperature β is characterized by the Kubo-Martin-Schwinger (KMS) condition [26]

$$\langle\rho;[\alpha_t^{(0)}A_1]A_2\rangle = \langle\rho;A_2\alpha_{t+i\hbar\beta}^{(0)}A_1\rangle\ \forall\ A_1,A_2\in\mathcal{A};\ t\in\mathbf{R}. \tag{2.4}$$

Most importantly, this condition survives the thermodynamic limit where N tends to infinity and the particle density ν remains finite [26]. Moreover, in this limit[e], the system may support different states that satisfy the condition. The set of these states is convex, and its extremal elements may naturally be interpreted as the pure equilibrium phases for the inverse temperature β [15, 29].

We assume that Σ has a linearly independent set of extensive conserved observables $\hat{Q} = (\hat{Q}_1,.\,.,\hat{Q}_n)$, which intercommute[f] up to surface effects and satisfy the following

[e] The model of the infinite system is formulated, in a standard way, in terms of its $C^{\star}$-algebra of quasi-local bounded observables [15, 26-28]. Its states are then positive normalized linear functionals on that algebra.

[f] The assumption of intercommutativity is not universally fulfilled. It is violated, for example in the case where $\hat{Q}_k$ and $\hat{Q}_l$, say, are different components of the magnetic moment of Σ. In such cases, some aspects of our treatment would have to be refined.

condition of *thermodynamical completeness* [15]:- *in the limit $N\to\infty$, the pure phases are labelled by, i.e are in one-to-one correspondence with, the expectation values $q_1,..,q_m$ of the global densities of $\hat{Q}_1,..,\hat{Q}_m$, respectively.* The resultant set of classical, intensive thermodynamical variables of Σ is then $q=(q_1,..,q_m)$. In general, we take $\hat{Q}_1$ to be the Hamiltonian of the system: correspondingly, q_1 is its energy density.

The equilibrium entropy density, in the limit $N\to\infty$, is a function, s, of q, which may be formulated by standard methods of quantum statistical mechanics [15, 27]. The classical equilibrium thermodynamics of the system is then governed by the form of $s(q)$. The demand of thermodynamical stability ensures that this function is concave. We define the thermodynamic conjugate of q_k to be $\theta_k=\partial s(q)/\partial q_k$. Thus, denoting the element $(\theta_1,..,\theta_m)$ of $\mathbf{R}^m$ by θ,

$$\theta = s'(q), \tag{2.5}$$

the derivative of $s(q)$, i.e. its gradient in q-space. Correspondingly, the second derivative, $s''(q)$, of this function is the Hessian $[\partial^2 s(q)/\partial q_k \partial q_l]$. We assume throughout this treatment that the system is in a single phase region, i.e. one where s is infinitely differentiable, where the function $q\to\theta(q)$ is invertible and where, for each value of q, the matrix $s''(q)$ is invertible. We define

$$J(q) := -s''(q)^{-1}, \tag{2.6}$$

which, in view of the concavity of s, is a positive matrix.

2.3. The Reservoir System $\mathcal{R}$. We assume that $\mathcal{R}$ comprises a set, $\{\mathcal{R}_J\}$, of spatially disjoint reservoirs, such that each $\mathcal{R}_J$ is placed in contact with a subregion $\partial\Omega_{N,J}$ of $\partial\Omega_N$ and $\bigcup_J \partial\Omega_{N,J}=\partial\Omega_N$. Further, we assume that each $\mathcal{R}_J$ has a thermodynamically complete set of global extensive conserved observables $(\hat{Q}_{J,1},..,\hat{Q}_{J,m})$ that are the natural counterparts of $\hat{Q}_1,..,\hat{Q}_m$, respectively, in that, when Σ and $\mathcal{R}_J$ are placed in contact, the observables $(\hat{Q}_k+\hat{Q}_{J,k})$ of $\Sigma^{(c)}$ are still conserved. Correspondingly, the thermodynamic control variables of $\mathcal{R}_J$ conjugate to Q_J are the same as those of Σ, namely θ. We denote by $\omega_J(\theta_J)$ the equilibrium state of $\mathcal{R}_J$ for which its θ-value is θ_J.

2.4. Nonequilibrium Steady States of $\Sigma^{(c)}$. Returning now to the situation where Σ is an open system, we assume that this is prepared according to the following prescription. Σ and the reservoirs $\{\mathcal{R}_J\}$ are independently prepared in the remote past in states ρ_0 and $\{\omega_J(\theta_J)\}$, respectively, where ρ_0 is normal and the values of θ_J generally varies from reservoir to reservoir: thus, in general, the reservoirs $\{\mathcal{R}_J\}$ are not in equilibrium with one another. Following this preparation the systems Σ and $\mathcal{R}$ are then coupled together and the resultant conservative composite evolves freely according to the dynamics governed by the automorphisms $\alpha(\mathbf{R})$. We assume that, as established under suitable asymptotically abelian conditions [12, 13], this dynamics acts so as to drive the system[g] $\Sigma^{(c)}$ into a terminal ρ_0-independent state $\phi(=w^\star-\lim_{t\to\infty}\alpha_t^\star[\rho_0\otimes_J\omega_J(\theta_J)]\ )$, whose restriction to $\mathcal{A}$ is

[g] The same result has been also obtained constructively [30] for certain models, which however are too rudimentary for our present purposes. In particular, the version of Σ there is just an multi-level atom.

normal. This state is uniquely determined by the states $\{\omega_J(\theta_J)\}$. Accordingly, we take ϕ to be the nonequilibrium steady state of $\Sigma^{(c)}$ stemming from the specified preparation, and we denote its GNS triple by $(\mathcal{H}_\phi, \pi, \Phi)$.

We note that, in view of the stationarity of ϕ, the automorphisms $\alpha(\mathbf{R})$ are implemented by a unitary representation U of $\mathbf{R}$ in $\mathcal{H}_\phi$ according to the prescription [31]

$$\pi(\alpha_t B) = U_t \pi(B) U_t^{-1} \ \forall \ B \in \mathcal{B}, \ t \in \mathbf{R}, \tag{2.7}$$

where U is defined by the formula

$$U_t \pi(B)\Phi = \pi(\alpha_t B)\Phi \ \forall \ B \in \mathcal{B}, \ t \in \mathbf{R}. \tag{2.8}$$

Since Eq. (2.8) is applicable to the subalgebra $\mathcal{A}$ of $\mathcal{B}$, the dynamics of the open system Σ, in the normal folium of ϕ, is given by the isomorphisms implemented by U of $\pi(\mathcal{A})$ into $\pi(\mathcal{B})$.

Moreover, this prescription extends to the unbounded observables of Σ for the following reasons. Since the restriction of ϕ to $\mathcal{A}$ is normal, so too, by Eq. (2.7), are the representations π and $\pi \circ \alpha_t$. It follows [32] that these representations have canonical extensions to the unbounded observables, S, of Σ according to the prescription that, if $\{E_\lambda\}$ is the family of spectral projectors of S, then those of $\pi(S)$ and $\pi(\alpha_t S)$ are $\{\pi(E_\lambda)\}$ and $\{\pi(\alpha_t E_\lambda) = U_t \pi(E_\lambda) U_t^{-1}\}$, respectively. Hence, the extension of the formula (2.7) to the unbounded observables takes the form

$$\pi(\alpha_t S) = U_t \pi(S) U_{-t} \tag{2.9}$$

for all unbounded observables S of Σ.

2.5. The Fields $\hat{q}$ and the Currents $\hat{j}$. We assume that, in the GNS representation π for the nonequilibrium steady state ϕ, the m-component extensive thermodynamical observable $\hat{Q}$ has a position-dependent, locally conserved density $\hat{q}(x) = \big(\hat{q}_1(x), . ., \hat{q}_m(x)\big)$, with associated current density $\hat{j}(x) = (\hat{j}_1(x), . ., \hat{j}_m(x))$. Thus the $\hat{q}_k$'s and $\hat{j}_k$'s are quantum fields and, in accordance with the general requirements of quantum field theory [25], we assume that they are distributions[h], in the sense of L. Schwartz [33].

We formulate these distributions in terms of the Schwartz spaces, $\mathcal{D}(\Omega_N)$ and $\mathcal{D}_V(\Omega_N)$, of real, infinitely differentiable scalar and $\mathbf{R}^d$-vector valued functions, respectively, on X with support in Ω_N. We define $\mathcal{D}^m(\Omega_N)$ and $\mathcal{D}_V^m(\Omega_N)$, respectively, to be the real vector spaces given by their m'th topological powers, equipped with the operations of binary addition and multiplication by real numbers given by the formula

$$\lambda(f_1, . ., f_m) + \lambda'(f_1', . \ , f_m') = (\lambda f_1 + \lambda' f_1', . ., \lambda f_m + \lambda' f_m')$$

[h] In concrete cases, it is a simple matter to verify that the explicit formulae for these fields and currents are indeed distributions. For example, the number density operator at position x is simply $\sum_{r=1}^N \delta(x - x_r)$, where x_r is the position of the r'th particle.

$$\forall\ \lambda, \lambda' \in \mathbf{R},\ f_k, f'_k \in \mathcal{D}(\Omega) \text{ or } \mathcal{D}_V(\Omega),\ k = 1, . ., m.$$

We denote by $\mathcal{D}'^m(\Omega_N)$ and $\mathcal{D}'^m_V(\Omega_N)$ the topological dual vector spaces of $\mathcal{D}^m(\Omega_N)$ and $\mathcal{D}^m_V(\Omega_N)$ respectively. Evidently, these are spaces of distributions (cf. [33])).

We assume that the m-component fields $\hat{q}(x)$ and $\hat{j}(x)$ are operator valued elements of $\mathcal{D}'^m(\Omega_N)$ and $\mathcal{D}'^m_V(\Omega_N)$, respectively. For simplicitly, we also assume that the components, $\hat{q}_k$, of $\hat{q}$ are invariant under time-reversals[i], i.e that they commute with the Wigner time reversal operator T.

The algebraic properties of the field $\hat{q}(x)$ are governed by the forms of the commutators $[\hat{q}_k(x), \hat{q}_l(y)]$. We assume that these take the following form, which is readily verified by the use of standard formulae in the case where $\hat{q}_1$ is the energy density of the system and the remaining $\hat{q}_k$'s are the particle number densities for the different species of its constituent particles.

$$[\hat{q}_k(x), \hat{q}_l(y)] = i\hbar \sum_{r=1}^{m} c_{klr} \hat{j}_r(x).\nabla\delta(x-y), \tag{2.10}$$

where the c's are N-independent constants. This formula evidently accords with our assumption that $\hat{Q}_k$'s intercommute, up to surface effects: indeed it implies that their commutators are just the integrals of currents over $\partial\Omega_N$.

We denote by $\hat{q}(f)$ and $\hat{j}(g)$ the 'smeared fields' obtained by integrating the distributions $\hat{q}$ and $\hat{j}$ against test functions $f = (f_1, . ., f_m)$ and $g = (g_1, . ., g_m)$, which belong to the spaces $\mathcal{D}^m(\Omega_N)$ and $\mathcal{D}^m_V(\Omega_N)$ respectively. Thus

$$\hat{q}(f) = \sum_{k=1}^{m} \int_{\Omega_N} dx \hat{q}_k(x) f_k(x) \tag{2.11}$$

and

$$\hat{j}(g) = \sum_{k=1}^{m} \int_{\Omega_N} dx \hat{j}_k(x).g_k(x). \tag{2.12}$$

In general, these smeared fields are unbounded observables, affiliated to the algebra $\mathcal{A}$. Therefore, by Eq. (2.7), their evolutes at time t, which we denote by $\hat{q}_t(f)$ and $\hat{j}_t(g)$, are their transforms implemented by the unitary operator U_t. Thus, they are the smeared fields corresponding to distribution valued operators $\hat{q}_t(x) = U_t\hat{q}(x)U_t^{-1}$ and $\hat{j}_t(x) = U_t\hat{j}(x)U_t^{-1}$, respectively; and the analogous statement may evidently be made for their components $\hat{q}_{k,t}(x)$ and $\hat{j}_{k,t}(x)$. For notational convenience, we shall sometimes denote $\hat{q}_t(x)$, $\hat{q}_t(f)$, $\hat{j}_t(x)$ and $\hat{j}_t(g)$ by $\hat{q}(x,t)$, $\hat{q}(f,t)$, $\hat{j}(x,t)$ and $\hat{j}(g,t)$, respectively.

We assume that the cyclic vector Φ for the state ϕ lies in the domain of all monomials in the smeared fields $\hat{q}_t(f)$ and $\hat{j}_{t'}(g)$ and that the resultant vector values of these monomials are continuous in the f's, g's, t's and t''s.

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[i] Standard examples of time-reversal invariant $\hat{q}_k$'s are the local number and energy densities of many-particle systems.

Since $\hat{j}$ is the current associated with $\hat{q}$, the local conservation laws for the latter field may be expressed in the form

$$\hat{q}_t(f)-\hat{q}_s(f)=\int_s^t du\hat{j}_u(\nabla f)\ \forall\ t,s\in\mathbf{R},\ f\in\mathcal{D}^m(\Omega_N). \tag{2.13}$$

2.6. The Hydrodynamical Scaling. We assume that the hydrodynamical observables of the open system Σ comprise just the m-component field $\hat{q}$, as viewed on the scale where the unit of length is L_N. Thus, on this scale, the system is confined to the fixed region Ω. Further, in accordance with our assumption, following Eq. (1.6), that the macroscopic dynamics is invariant under space-time scale transformations $x{\rightarrow}\lambda x,\ t{\rightarrow}\lambda^2 t$, we assume that L_N^2 is the unit of time corresponding to the length unit L_N. Hence, in the normal folium of the nonequilibrium steady state ϕ, the m-component hydrodynamic field is represented by the distribution valued operator

$$\check{q}_t(x):=\hat{q}(L_Nx,L_N^2t). \tag{2.14}$$

It follows from this equation and Eq. (2.11) that the smeared hydrodynamic field obtained by integrating $\check{q}_t(x)$ against a $\mathcal{D}^m(\Omega)$-class test function f is

$$\check{q}_t(f)=\hat{q}(f^{(N)},L_N^2t),\ \forall\ f{\in}\mathcal{D}^m(\Omega),\ t{\in}\mathbf{R}, \tag{2.15}$$

where $f^{(N)}$ $({\in}\mathcal{D}^m(\Omega_N))$ is related to f according to the formula

$$f^{(N)}(x)=L_N^{-d}f(L_N^{-1}x)\ \forall\ x{\in}\Omega_N. \tag{2.16}$$

Since the scale transformation $(x,t){\rightarrow}(L_Nx,L_N^2t)$ sends $\hat{q}$ to $\check{q}$, it follows that the local conservation law (2.13), or formally $\partial\hat{q}_t(x)/\partial t=-\nabla.\hat{j}_t(x)$, will be preserved if its sends $\hat{j}_t(x)$ to $\check{j}_t(x)$, where

$$\check{j}_t(x):=L_N\hat{j}(L_Nx,L_N^2t). \tag{2.17}$$

It follows from this formula and Eq. (2.12) that the smeared field obtained by integrating $\check{j}_t(x)$ against a $\mathcal{D}_V^m(\Omega)$- class test function g is

$$\check{j}_t(g)=\hat{j}(g^{(N)},L_N^2t), \tag{2.18}$$

where

$$g^{(N)}(x)=L_N^{1-d}g(L_N^{-1}x). \qquad (2.19).$$

In view of Eqs. (2.15) and (2.18), it is a simple matter to confirm that the local conservation law (2.13) retains its form in the macroscopic description, i.e. that

$$\check{q}_t(f)-\check{q}_s(f)=\int_s^t du\check{j}_u(\nabla f)\ \forall\ t,s\in\mathbf{R},\ f\in\mathcal{D}^m(\Omega). \tag{2.20}$$

3. Connection between the Quantum Picture, the Phenomenological Dynamics and the Hydrodynamical Fluctuations

We now seek an inter-relationship between the quantum and hydrodynamical properties of the macroscopic field $\check{q}_t(x)$ and its current $\check{j}_t(x)$ in the limit where N tends to infinity. In order to formulate this limit, we shall henceforth indicate the N-dependence of the quantum model by attaching the superscript (N) to the symbols $\Sigma,\ \phi,\ \Phi,\ U,\ \hat{q}, \hat{j},\ \check{q}$ and $\check{j}$. The symbol Σ, without that superscript, will be reserved for the limiting case where N becomes infinite. The symbol Ω, on the other hand, will continue to represent the fixed region occupied by $\Sigma^{(N)}$, in the *hydrodynamical* scaling, for all N.

Our basic assumptions concerning the relationship between the quantum and hydrodynamic pictures of the model are that, in the limit $N\to\infty$,

(a) the stationary hydrodynamic fields $q(x)$ and $j(x)$ are the expectation values of the quantum fields $\check{q}_t^{(N)}(x)$ and $\check{j}_t^{(N)}(x)$, respectively, for the steady state $\phi^{(N)}$; and

(b) the regressions of the fluctuations of these fields are governed, in a sense that will be made precise in Section 4, by the same dynamical laws (1.7) and (1.8) as the weak perturbations $\delta q_t(x)$ and $\delta j_t(x)$ of $q(x)$ and $j(x)$, respectively.

The regression hypothesis (b) is a natural generalization of that proposed by Onsager [17] for fluctuations about equilibrium states. We remark here that, since $\mathcal{D}'$ spaces are complete, these assumptions imply that the classical fields $q(x),\ j(x),\ \delta q_t(x)$ and $\delta j_t(x)$, introduced in Section 1, are distributions.

3.1. Quantum Statistical Formulae the Hydrodynamical Variables. It follows immediately from our specifications that the above assumption (a) signifies that

$$q(x) = \lim_{N\to\infty}\bigl(\Phi^{(N)}, \check{q}_t^{(N)}(x)\Phi^{(N)}\bigr) \tag{3.1}$$

and

$$j(x) = \lim_{N\to\infty}\bigl(\Phi^{(N)}, \check{j}_t^{(N)}(x)\Phi^{(N)}\bigr), \tag{3.2}$$

the t-independence of the r.h.s.'s of these formula being guaranteed by the stationarity of $\phi^{(N)}$.

In order to bring the hydrodynamical description of the model into line with thermodynamics, we introduce the field $\theta(x) = \bigl(\theta_1(x), .\ , \theta_m(x)\bigr)$, conjugate to $q(x)$ as defined by the space-dependent version of Eq. (2.5), namely

$$\theta(x) = s'\bigl(q(x)\bigr). \tag{3.3}$$

Since we are assuming that the system is perpetually in a single phase region, and thus that the function s' is invertible, it follows from this formula that the fields $q(x)$ and $\theta(x)$ are in one-to-one correspondence.

Turning now to the hydrodynamical equation (1.4), we see immediately that the stationary field $q(x)$ is determined by the requirement that $\mathcal{F}(q;x) = 0$, together with the

conditions imposed by the $\Sigma^{(N)}-\mathcal{R}$ coupling at the boundary $\partial\Omega$ of Ω. In order to specify these conditions, we denote by $\partial\Omega_J$ the section of $\partial\Omega$ where $\Sigma^{(N)}$ is in contact with $\mathcal{R}_J$. We then assume the following boundary condition.

(**R**) On the section $\partial\Omega_J$ of the boundary of Σ, the classical field $\theta(x)$ of this system takes the value θ_J of the control variables of the equilibrium state in which $\mathcal{R}_J$ is initially prepared. Thus the array of reservoirs fixes the form of $\theta(x)$ and therefore of $q(x)$ on $\partial\Omega$.

This assumption signifies that, on the *hydrodynamic* time scale and in the limit $N\to\infty$, the local thermodynamical variables $\theta(x)$ of Σ spontaneously take up the same values as the reservoir with which this system is in contact at its boundary. The assumption is fulfilled by the models of Refs. [9-11].

Note on the Phenomenological Dynamics: $\nabla\theta$ as Driving Force. In the general situation where the field q_t is time-dependent, we define its thermodynamical conjugate to be the field θ_t given by the space-time dependent version of Eq. (2.5), namely

$$\theta_t(x)=s'\big(q_t(x)\big). \tag{3.4}$$

Thus, in view of our assumption that the system is perpetually in a single phase region, the function s' is invertible and the phenomenological law (1.4) may be expressed in the form

$$\frac{\partial}{\partial t}q_t(x)=\nabla.\mathcal{G}\big(\theta_t;x\big), \tag{3.5}$$

where the functional $\mathcal{G}$ is determined by $\mathcal{J}$ according to the formula

$$\mathcal{G}\big(s'(q_t);x\big)=-\mathcal{J}(q_t;x). \tag{3.6}$$

In particular, in the case of nonlinear diffusion, it follows from Eqs. (1.4), (1.5), (2.5) and (2.6) that this phenomenological law reduces to the form

$$\frac{\partial}{\partial t}q_t(x)+\nabla.\Big(K\big(\theta_t(x)\big)\nabla\theta_t(x)\Big)=0, \tag{3.7}$$

where, in correspondence with the general relationship (2.5) between q and θ,

$$K(\theta)=\tilde{K}(q)J(q)\equiv\tilde{K}\big([s']^{-1}(\theta)J\big([s']^{-1}(\theta)\big). \tag{3.8}$$

One sees immediately from Eq. (3.7) that the gradient of the thermodynamical field θ_t acts as the hydrodynamical driving force.

3.2. Linearized Perturbations of the Hydrodynamics. In view of our above remarks, δq_t is a distribution that satisfies Eq. (1.7) and vanishes on $\partial\Omega$. We assume that the linear operator $\mathcal{L}$ appearing in that equation is the generator of a one-parameter semigroup, $\{T_t|t\in\mathbf{R}_+\}:=T(\mathbf{R}_+)$, of transformations of $\mathcal{D}'^m(\Omega)$. The solution of Eq. (1.7) is then

$$\delta q_t=T_{t-s}\delta q_s\ \forall\ t\geq s\geq 0. \tag{3.9}$$

Correspondingly, by Eq. (1.8),

$$\delta j_t = \mathcal{K}\delta q_t = \mathcal{K}T_{t-s}\delta q_s \ \forall \ t{\geq}s. \tag{3.10}$$

Further, by Eq. (3.9) and the dissipativity condition stated in the paragraph before Eq. (1.7),

$$\mathcal{D}' - \lim_{t\to\infty} T_t\psi = 0 \ \forall \ \psi{\in}\mathcal{D}'^m(\Omega) \tag{3.11}$$

or equivalently

$$\mathcal{D} - \lim_{t\to\infty} T_t^\star f = 0 \ \forall \ f{\in}\mathcal{D}^m(\Omega), \tag{3.12}$$

where $\{T_t^\star|t{\in}\mathbf{R}_+\}$ is the one-parameter semigroup of transformations of $\mathcal{D}^m(\Omega)$ dual to $T(\mathbf{R}_+)$. We denote its generator by $\mathcal{L}^\star$, which is just the dual of $\mathcal{L}$.

3.3. The Hydrodynamical Fluctuation Fields. We define the quantum fields, $\xi_t^{(N)}(x) = \big(\xi_{1,t}^{(N)}(x), .. , \xi_{m,t}^{(N)}(x)\big)$ and $\eta_t^{(N)} = \big(\eta_{1,t}^{(N)}(x), .. , \eta_{m,t}^{(N)}(x)\big)$, representing the fluctuations of the hydrodynamically scaled field $\check{q}_t^{(N)}(x)$ and the associated current $\check{j}_t^{(N)}(x)$, by the formulae

$$\xi_t^{(N)}(x) = L_N^{d/2}\big[\check{q}_t^{(N)}(x) - \big(\Phi^{(N)}, \check{q}_t^{(N)}(x)\Phi^{(N)}\big)\big], \tag{3.13}$$

and

$$\eta_t^{(N)}(x) = L_N^{d/2}\big[\check{j}_t^{(N)}(x) - \big(\Phi^{(N)}, \check{j}_t^{(N)}(x)\Phi^{(N)}\big)\big], \tag{3.14}$$

the normalization factor $L_N^{d/2}$ being natural for this scaling. The corresponding smeared fields $\xi_t^{(N)}(f)$ and $\eta_t^{(N)}(g)$ are then the observables obtained by integrating these fields against test functions f $({\in}\mathcal{D}^m(\Omega))$ and g $({\in}\mathcal{D}_V^m(\Omega))$, respectively. Thus, it follows from Eqs. (2.20), (3.13) and (3.14) that $\xi_t^{(N)}$ satisfies the local conservation law

$$\xi_t^{(N)}(f) - \xi_s^{(N)}(f) = \int_s^t du\eta_u^{(N)}(\nabla f) \ \forall \ t, s \in \mathbf{R}, \ f \in \mathcal{D}^m(\Omega_N). \tag{3.15}$$

The dynamical properties of the fluctuation field $\xi_t^{(N)}$ are encoded in the correlation functions

$$W^{(N)}(f^{(1)}, . ., f^{(r)}; t_1, . ., t_r) = \big(\Phi^{(N)}, \xi_{t_1}^{(N)}(f^{(1)}). .\xi_{t_r}^{(N)}(f^{(r)})\Phi^{(N)}\big). \tag{3.16}$$

This formula, together with Eqs. (2.15) and (3.13), serves to express $W^{(N)}$ in terms of the smeared fields $\hat{q}_t^{(N)}(f)$ of Section 2. Thus, in view of our stipulation there that the actions on $\Phi^{(N)}$ of the monomials in these fields are continuous in the f's, and t's, it follows that $W^{(N)}$ is continuous in all its arguments. Further, it follows from the stationarity of the state $\phi^{(N)}$ and the self-adjointness of the observables $\xi_t^{(N)}(f)$ that

$$W^{(N)}(f^{(1)}, . ., f^{(r)}; t_1 + a, . ., t_r + a) = W^{(N)}(f^{(1)}, . ., f^{(r)}; t_1, . ., t_r) \ \forall \ a{\in}\mathbf{R}, \tag{3.17}$$

and

$$\overline{W}^{(N)}(f^{(1)}, . ., f^{(r)}; t_1, . ., t_r) = W^{(N)}(f^{(r)}, . ., f^{(1)}; t_r, . ., t_1); \tag{3.18}$$

while the positivity of $\phi^{(N)}$ implies that $(A\Phi^{(N)},A\Phi^{(N)})\geq 0$ for any polynomial A in the smeared fields $\xi_t^{(N)}(f)$. Thus choosing $A=\sum_{k=1}^p c_k\xi_{t_{k,1}}^{(N)}(f^{(k,1)}).\,.\xi_{t_{k,r_k}}^{(N)}(f^{(k,r_k)})$, where the c's are complex constants and p is finite,

$$\sum\nolimits_{k,l=1}^{p}\overline{c}_k c_l W^{(N)}(f^{(k,r_k)},.\,.,f^{(k,1)}f^{(l,1)},.\,.,f^{(l,r_l)};t_{k,r_k},.\,.,t_{k,1},t_{l,1},.\,.,t_{l,r_l})\geq 0. \qquad (3.19)$$

3.4. Hydrodynamic Limit of the Fluctuation Process. We now assume that $W^{(N)}$ converges to a functional W in the hydrodynamic limit where $N\to\infty$, i.e. that

$$\lim_{N\to\infty}W^{(N)}(f^{(1)},.\,.,f^{(r)};t_1,.\,.,t_r)=W(f^{(1)},.\,.,f^{(r)};t_1,.\,.,t_r)$$

$$\forall\ f^{(1)},.\,.,f^{(r)}\in\mathcal{D}^m(\Omega),\ t_1,.\,.,t_r\in\mathbf{R},\ r\in\mathbf{N}. \qquad (3.20)$$

Hence, in view of the continuity properties of $W^{(N)}$ and the completeness of $\mathcal{D}'$ spaces, W is continuous in the f's and measurable in the t's. It is therefore a zero order distribution with respect to the latter variables [33]. Further, it follows immediately from Eq. (3.20) that W inherits the stationarity, Hermiticity and positivity properties of $W^{(N)}$, as given by Eqs. (3.17)-(3.19). Thus

$$W(f^{(1)},.\,.,f^{(r)};t_1+a,.\,.,t_r+a)=W(f^{(1)},.\,.,f^{(r)};t_1,.\,.,t_r)\ \forall\ a\in\mathbf{R}, \qquad (3.21)$$

$$\overline{W}(f^{(1)},.\,.,f^{(r)};t_1,.\,.,t_r)=W(f^{(r)},.\,.,f^{(1)};t_r,.\,.,t_1); \qquad (3.22)$$

and

$$\sum\nolimits_{k,l=1}^{p}\overline{c}_k c_l W(f^{(k,r_k)},.\,.,f^{(k,1)},f^{(l,1)},.\,.,f^{(l,r_l)};t_{k,r_k},.\,.,t_{k,1},t_{l,1},.\,.,t_{l,r_l})\geq 0. \qquad (3.23)$$

It follows from these properties that, by Wightman's reconstruction theorem [25], W corresponds precisely to the quadruple $(\mathbf{H},V,\xi,\Psi)$, where

(a) $\mathbf{H}$ is a Hilbert space,

(b) V is a unitary representation of $\mathbf{R}$ in $\mathbf{H}$ such that V_t, the image of t ($\in\mathbf{R}$) under V, is strongly measurable;

(c) $\xi_t(x)$ is a Hermitian operator valued distribution, of class $\mathcal{D}'^m(\Omega)$, in $\mathbf{H}$, which implements the time translations of ξ, i.e.

$$\xi_{t+s}(x)=V_t\xi_s(x)V_t^{-1}; \qquad (3.24)$$

and

(d) Ψ is a vector in $\mathbf{H}$ that is invariant under V_t and cyclic with respect to the polynomials in the smeared fields $\xi_t(f)$ obtained by integrating $\xi_t(x)$ against $\mathcal{D}^m(\Omega)$-class test functions f.

The functional W is then related to the smeared field $\xi_t(x)$ and the cyclic vector Ψ by the formula

$$W(f^{(1)},..,f^{(r)};t_1,..,t_r)=\big(\Psi,\xi_{t_1}(f^{(1)})..\xi_{t_r}(f^{(r)})\Psi\big). \tag{3.25}$$

3.5. Conditions for W to represent a Classical Stochastic Process. The question of whether W represents a classical stochastic process reduces to those of whether (a) it defines a quantum stochastic process in the sense of Ref. [34] and (b) this process has the abelian properties of a classical one. Now the condition (a) is fulfilled if the smeared Hermitian fields $\xi_t(f)$ are self-adjoint since, in this case, the unitary operators $\{\exp\big(i\xi_t(f)\big)|f{\in}\mathcal{D}^m(\Omega)\}$ generate a $W^{\star}$-algebra $\mathcal{N}_t$ and the correlation functions $\big\{(\Psi,F_{t_1}..F_{t_r}\Psi)|F_{t_s}{\in}\mathcal{N}_{t_s};\ s=1,..,r\big\}$ define a quantum stochastic process, as formulated in [34]. Further, the classicality condition[j] (b) is simply that of the intercommutativity of the operators $\xi_t(f)$.

The following proposition provides a sufficient condition for the functional W to represent a quantum stochastic process.

Proposition 3.1. *The functional W uniquely defines a quantum stochastic process ξ, indexed by $\mathcal{D}^m(\Omega){\times}\mathbf{R}$, if there is a bounded, positive functional $(f,t){\to}F_t(f)$ on that product space such that*

$$|W(f^{(1)},..,f^{(r)};t_1,..,t_r)|{\le}r^2F_{t_1}(f^{(1)})..F_{t_r}(f^{(r)})\ \forall\ f^{(1)},..,f^{(r)}\ {\in}\mathcal{D}^m(\Omega);\ t_1,..,t_r{\in}\mathbf{R}. \tag{3.26}$$

Comment. We shall subsequently establish in Prop. 6.1 that, under the assumptions of our scheme, the process ξ is Gaussian. Since that result implies that the truncated r-point functions induced by W all vanish and thus that Eq. (3.26) is satisfied, it signifies a consistency of our assumptions.

Proof of Prop. 3.1. As noted above, W defines a stochastic process if the Hermitian operators $\xi_t(f)$ are self-adjoint; and by Nelson's theorem [35], a sufficient condition for this is that each of these fields has a dense domain of analytic vectors. To prove that this is the case, subject to the assumption of Eq. (3.26), we note that it follows from that inequality and Eq. (3.25) that, for arbitrary $f,\ f^{(1)},..,f^{(r)}$ in $\mathcal{D}^m(\Omega)$ and $t,\ t_1,..,t_r$ in $\mathbf{R}$,

$$\|\xi_t(f)^p\xi_{t_1}(f^{(1)})...\xi_{t_r}(f^{(r)})\Psi\|{\le}(p+r)^2F_t(f)^pF_{t_1}(f^{(1)})...F_{t_r}(f^{(r)})$$

and therefore that the $\mathbf{H}$-valued function $z({\in}\mathbf{C}){\to}\sum_{p=0}^{\infty}z^p\xi_t(f)^p\xi_{t_1}(f^{(1)})..\xi_{t_r}(f^{(r)})\Psi/p!$ has an infinite radius of convergence. Hence, in view of the cyclicity of Ψ with respect to the polynomials in the smeared fields $\{\xi_t(f)\}$, these fields are self-adjoint and therefore W corresponds to a stochastic process.

We shall assume henceforth that W does indeed define a stochastic process. In order to formulate a condition for its classicality, we introduce the following definition.

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[j] Here we consider classical processes as special (abelian) cases of the quantum ones.

Definition 3.2. (1) We define $\mathcal{P}$ (resp. $\mathcal{P}^{(N)}$) to be the set of polynomials in the smeared fields $\{\xi_t(f)\ \big(\text{resp. } \xi_t^{(N)}(f)\big) | f{\in}\mathcal{D}^m(\Omega),\ t{\in}\mathbf{R}\}$ and we define the bijection $P{\rightarrow}P^{(N)}$ of $\mathcal{P}$ onto $\mathcal{P}^{(N)}$ by the prescription that $P^{(N)}$ is the element of $\mathcal{P}^{(N)}$ obtained by replacing ξ by $\xi^{(N)}$ in the formula for P.

(2) For $P{\in}\mathcal{P}$ and $N{\in}\mathbf{N}$, we define the vector $\Psi_P^{(N)}$ $({\in}\mathcal{H}_{\phi^{(N)}})$ by the formula

$$\Psi_P^{(N)} = P^{(N)}\Phi^{(N)}. \tag{3.27}$$

We now note that, by Eq. (3.25), the classicality condition that the operators $\xi_t(f)$ intercommute is equivalent to the invariance of $W(f^{(1)},.\ ,f^{(k)};t_1,.\ ,t_n)$ under the permutations

$$(f^{(r)},t_r){\rightleftharpoons}(f^{(r+1)},t_{r+1});$$

and by Def. (3.2) and Eqs. (3.12), (3.16), (3.20), this latter condition may be expressed in the form

$$\lim_{N\to\infty}\big(\Psi_P^{(N)},[\xi_t^{(N)}(f),\xi_{t'}^{(N)}(f')]\Psi_P^{(N)}\big) = 0$$
$$\forall\ P{\in}\mathcal{P},\ f,f'{\in}\mathcal{D}^m(\Omega),\ t,t'{\in}\mathbf{R}.$$

Moreover, we can set $t'=0$ here without loss of generality, since $\Phi^{(N)}$ is invariant under $U_t^{(N)}$ and therefore, by Eq. (2.14), Def. 3.2 and the definition of $\xi_t^{(N)}$, the manifold $\mathcal{P}^{(N)}\Phi^{(N)}$ is stable under this unitary transformation. Consequently, we have the following proposition, whose significance we shall discuss below.

Proposition 3.3. *Under the above assumptions, the process ξ is classical if and only if $\xi_t^{(N)}(f)$ satisfies the condition that*

$$\lim_{N\to\infty}\big(\Psi_P^{(N)},[\xi_t^{(N)}(f),\xi^{(N)}(f')]\Psi_P^{(N)}\big) = 0\ \forall P{\in}\mathcal{P},\ f,f'{\in}\mathcal{D}^m(\Omega),\ t{\in}\mathbf{R}. \tag{3.28}$$

Comment. In order to relate the condition (3.28) to the microscopic picture, we infer from Eqs. (2.10), (2.14)-(2.19) and (3.13) that this condition signifies the following.

(1) In the case where $t{\neq}0$,

$$\lim_{N\to\infty}L_N^d\sum_{k,l=1}^m\int_{\Omega^2}dxdy\big(\Psi_P^{(N)},[\hat{q}_k(L_Nx,L_N^2t),\hat{q}_l(L_Ny)]\Psi_P^{(N)}\big)f_k(x)f_l'(y) = 0$$
$$\forall\ f,f'{\in}\mathcal{D}^m(\Omega),\ P{\in}\mathcal{P}, \tag{3.29}$$

which is evidently a space-time asymptotic abelian condition on the field $\hat{q}$.

(2) In the case where $t=0$,

$$\lim_{N\to\infty}L_N^{-2}\big(\Psi_P^{(N)},\check{j}^{(N)}(g_{f,f'})\Psi_P^{(N)}\big) = 0$$

$$\forall\ f,f'\in\mathcal{D}^m(\Omega),\ \ P\in\mathcal{P}, \tag{3.30}$$

where $g_{f,f'}$ is the element of $\mathcal{D}_V^m(\Omega)$ whose rth component is

$$g_{f,f';r}=\sum\nolimits_{kl}\hbar c_{rkl}f_k\nabla f'_l. \tag{3.31}$$

Thus, Eq. (3.30) signifies the avoidance of the catastrophe whereby, for fixed $P\in\mathcal{P}$, the expectation value of the smeared hydrodynamically scaled current $\check{j}^{(N)}(g_{f,f'})$ in the vector state $\Psi_P^{(N)}$ would grow as rapidly as L_N^2 with increasing N.

4. The Stochastic Process ξ: Regression and Local Equilibrium Hypotheses and the Generalized Onsager Relations

We now assume that the conditions of Props. 3.1 and 3.3 are fulfilled and hence that ξ is a classical stochastic process, indexed by $\mathbf{R}\times\mathcal{D}^m(\Omega)$. In a standard way, we denote the expectation functional of the random variables for this process by E. Thus, by Eq. (3.25),

$$E\big(\xi_{t_1}(f^{(1)}).\,.\xi_{t_r}(f^{(r)})\big)=\big(\Psi,\xi_{t_1}(f^{(1)}).\,.\xi_{t_r}(f^{(r)})\Psi\big)\ \forall\ t_1,.\,.,t_r\in\mathbf{R},\ f^{(1)},.\,.,f^{(r)}\in\mathcal{D}^m(\Omega). \tag{4.1}$$

We note that, by Eqs. (3.20), (3.25) and (4.1), the process $\xi^{(N)}$ converges to ξ, i.e. its correlation functions converge to the corresponding ones for ξ, as $N\to\infty$. Further, in view of the observation following Eq. (3.20), the correlation function $E\big(\xi_{t_1}(f^{(1)}).\,.\xi_{t_r}(f^{(r)})\big)$ is continuous with respect to the f's and measurable with respect to the t's.

Conditional Expectations. For any random variable F of the ξ-process and for $t\in\mathbf{R}$, we denote the conditional expectations of F with respect to the σ-algebras generated by $\{\xi_t(f)|f\in\mathcal{D}^m(\Omega))\}$ and $\{\xi_{t'}(f)|t'\leq t,\ f\in\mathcal{D}^m(\Omega)\}$ by $E(F|\xi_t)$ and $E(F|\xi_{\leq t})$, respectively.

4.1. The Regression Hypothesis. This hypothesis is just the canonical generalization of that assumed by Onsager [17] for fluctuations about equilibrium states. Its essential import is that the evolution of a small hydrodynamical deviation from a steady state does not depend on whether the deviation has arisen from a spontaneous fluctuation or from a weak perturbation of the system[k]. Thus, in mathematical terms, the regression hypothesis asserts that, for fixed s and $t\geq s$, the evolution of $E(\xi_t|\xi_s)$ is governed by the same law as that of the linearised perturbation δq_t of the deterministic trajectory q_t, i.e., by Eq. (3.9), that

$$E\big(\xi_t(f)|\xi_s\big)=[T_{t-s}\xi_s](f)\equiv\xi_s(T^\star_{t-s}f)\ \forall\ t\geq s. \tag{4.2}$$

Hence, since Nelson's forward time derivative [36] of $\xi_t(f)$ is defined to be

$$D\xi_t(f):=\lim\nolimits_{u\to+0}u^{-1}E\big(\xi_{t+u}(f)-\xi_t(f)|\xi_t\big) \tag{4.3}$$

[k] As in Onsager's theory, the assumption of this equivalence between the consequences of fluctuations and weak perturbations is not quite innocuous, since the modifications of the variables q due to the former are $O(N^{-1/2})$, whereas those due to the latter are of order of a different small parameter that represents the strength of the perturbation.

and, since $\mathcal{L}$ is the generator of $T(\mathbf{R}_+)$, it follows that

$$D\xi_t(f) = \mathcal{L}\xi_t(f). \tag{4.4}$$

Further, defining the static two-point function $W_S : \mathcal{D}^m(\Omega)\times\mathcal{D}^m(\Omega)\rightarrow\mathbf{R}$ by the formula

$$W_S(f,f') = E\big(\xi(f)\xi(f')\big) \ \forall\ f,f'\in\mathcal{D}^m(\Omega), \tag{4.5}$$

it follows from Eq. (4.2) and the stationarity of the ξ- process that

$$E\big(\xi_t(f)\xi_{t'}(f')\big) = W_S(T^{\star}_{t-t'}f,f') \ \forall\ f,f'\in\mathcal{D}^m(\Omega),\ t,t'(\leq t)\in\mathbf{R}. \tag{4.6}$$

4.2. Local Equilibrium Conditions. Our next assumption asserts essentially that, in a nonequilibrium steady state, the statistical properties of the fluctuation field ξ in a 'small' neighbourhood, $\mathcal{N}(x)$, of an arbitrary point x ($\in\Omega$) simulate those enjoyed by these fields in the true equilibrium state corresponding to the value $q(x)$ of the thermodynamic variable q. This is a mesoscopic local equilibrium condition, since it involves only the fluctuation field ξ and is thus weaker than that of microscopic local equilibrium [37], which would signify that the microstate of Σ in $\mathcal{N}(x)$ simulated the equilibrium microstate corresponding to $q(x)$ there. Here we note that even this stronger condition has been shown to be fulfilled [38] by systems of fermions for which an Eulerian hydrodynamics has been established. Moreover, it may be expected to ensue more generally from the fact that the ratio of the hydrodynamic time scale to that of the microscopic processes (collisions etc.) is infinite, since that implies that local values of the hydro- thermodynamic variables q change negligibly in the time taken for the latter processes to generate equilibrium in macroscopically small spatial regions.

In order to precisely specify our mesoscopic local equilibrium hypothesis, we start by formulating the relevant properties of hydrodynamical fluctuations about true equilibrium states for which the stationary classical field $q(x)$ is assumed to be uniform.

Equilibrium Fluctuations. We recall that, for a *finite* system, the equilibrium probability distribution function, P, for macroscopic observables A is determined by the entropy $S(A)$ according to the Einstein formula

$$P(A) = \text{const.}\exp\big(S(A)\big),$$

and this serves to relate the static correlation functions for the fluctuations of these observables to the thermodynamics of the system. The generalization of this relation to infinite systems has been derived by a quantum statistical treatment [15, Ch.7, Appendix C] of equilibrium states and takes the form

$$E_{eq}\big(\xi(f)\xi(f')\big) = \big(f,J(q)f'\big),\ \forall\ f,f' \in \mathcal{D}^m(\Omega), \tag{4.7}$$

where E_{eq} is the equilibrium expectation functional for the fluctuation process, $J(q)$ is defined by Eq. (2.6) and $(.,.)$ is the inner product on $\mathcal{D}^m(\Omega)$ defined by the formula

$$(f,f') = \sum\nolimits_{k=1}^{m} \int_\Omega dx f_k(x)f'_k(x). \tag{4.8}$$

It follows from Eqs. (4.2) and (4.7) that

$$E_{eq}\big(\xi_t(f)\xi_s(f')\big) = \big(T^\star_{t-s}f, J(q)f'\big)\ \forall\ f, f' \in \mathcal{D}^m(\Omega),\ t{\geq}s. \tag{4.9}$$

Further, recalling the assumption, introduced in Section 2.5, of the invariance of the quantum field $\hat{q}^{(N)}(x)$ under the time-reversal antiautomorphism τ and assuming that the equilibrium state[l] $\phi^{(N)}_{eq}$ of $(\Sigma^{(N)} + \mathcal{R})$ is likewise time-reversal invariant, it follows from the stationarity of this state and Eq. (3.13) that

$$\langle \phi^{(N)}_{eq}; \xi^{(N)}_t(f)\xi^{(N)}(f')\rangle = \langle \phi^{(N)}_{eq}; \xi^{(N)}(f')\xi^{(N)}_{-t}(f)\rangle = \langle \phi^{(N)}_{eq}; \xi^{(N)}_t(f')\xi^{(N)}(f)\rangle.$$

On passing to the limit of this equation as $N{\rightarrow}\infty$, we see that

$$E_{eq}\big(\xi_t(f)\xi(f')\big) = E_{eq}\big(\xi_t(f')\xi(f)\big);$$

and therefore, by Eq. (4.9), that

$$E_{eq}\big(\xi(T^\star_t f)\xi(f')\big) = E_{eq}\big(\xi(T^\star_t f')\xi(f),\ \forall\ t{\geq}0.$$

Consequently, since $\mathcal{L}^\star$ is the generator of $T^\star(\mathbf{R}_+)$,

$$E_{eq}\big(\xi(\mathcal{L}^\star f)\xi(f')\big) = E_{eq}\big(\xi(\mathcal{L}^\star f')\xi(f)\big)\ \forall\ f, f'{\in}\mathcal{D}^m(\Omega). \tag{4.10}$$

Local Form of Equilibrium Correlations. We formulate the local properties of the equilibrium fluctuations in terms of test functions that are highly localised around an arbitrary point x_0 of Ω. Specifically, for $f{\in}\mathcal{D}^m(\Omega)$, $x_0{\in}\Omega$ and $\epsilon{\in}\mathbf{R}_+$, we define the function $f_{x_0,\epsilon}$ on the Euclidean space X by the formula

$$f_{x_0,\epsilon}(x) = \epsilon^{-d/2} f\big(\epsilon^{-1}(x - x_0)\big)\ \forall\ x_0{\in}\Omega,\ f{\in}\mathcal{D}^m(\Omega). \tag{4.11}$$

Since Ω is a bounded open subregion of X, it follows that the restriction of $f_{x_0,\epsilon}$ to Ω belongs to the space $\mathcal{D}^m(\Omega)$ for sufficiently small ϵ. In this case, we may take Eq. (4.11) to define a transformation $f{\rightarrow}f_{x_0,\epsilon}$ of $\mathcal{D}^m(\Omega)$, with ϵ representing the degree of localization of the latter function about the point x_0.

We now note that, by Eqs (4.8) and (4.11), the r.h.s. of Eq. (4.7) is invariant under the transformation $f{\rightarrow}f_{x_0,\epsilon}$ and therefore it follows from that equation that the equilibrium fluctuations enjoy the *local* property given by the formula

$$\lim_{\epsilon\downarrow 0} E_{eq}\big(\xi(f_{x_0,\epsilon})\xi(f'_{x_0,\epsilon})\big) = (f, J(q)f')\ \forall\ x_0{\in}\Omega,\ f, f'{\in}\mathcal{D}^m(\Omega). \tag{4.12}$$

Further, in the case of nonlinear diffusion, it follows from Eq. (1.10) that, for perturbations of the equilibrium state, $\mathcal{L} = \tilde{K}(q)\Delta$, with q constant. Hence, for fluctuations about

[l] The same assumption would not be valid for nonequilibrium states, since these generally carry currents of odd parity with respect to time reversals.

equilibrium, it follows from Eq. (4.7) that both sides of Eq. (4.10) are invariant under the transformation $f \to f_{x_0,\epsilon}, \ f' \to f'_{x_0,\epsilon}, \ E_{eq} \to \epsilon^2 E_{eq}$, and consequently

$$\lim_{\epsilon\downarrow 0}\epsilon^2 E_{eq}\big(\xi(\mathcal{L}^\star f_{x_0,\epsilon})\xi(f'_{x_0,\epsilon})\big) = \lim_{\epsilon\downarrow 0}\epsilon^2 E_{eq}\big(\xi(\mathcal{L}^\star f'_{x_0,\epsilon})\xi(f_{x_0,\epsilon}\big) \ \forall x_0\in\Omega. \tag{4.13}$$

Local Equilibrium Conditions for Nonequilibrium Steady States. We now assume that, for these states, the natural counterparts of the local conditions (4.12) and (4.13) still hold, i.e. that

$$\lim_{\epsilon\downarrow 0}E\big(\xi(f_{x_0,\epsilon})\xi(f'_{x_0,\epsilon})\big) = \big(f, J(q(x_0))f'\big) \ \forall \ x_0\in\Omega, \ \ f, f'\in\mathcal{D}^m(\Omega) \tag{4.14}$$

and

$$\lim_{\epsilon\downarrow 0}\epsilon^2 E\big(\xi(\mathcal{L}^\star f_{x_0,\epsilon})\xi(f'_{x_0,\epsilon})\big) = \lim_{\epsilon\downarrow 0}\epsilon^2 E\big(\xi(\mathcal{L}^\star f'_{x_0,\epsilon})\xi(f_{x_0,\epsilon})\big) \ \forall \ x_0\in\Omega, \ \ f, f'\in\mathcal{D}^m(\Omega). \tag{4.15}$$

These are our local equilibrium conditions, which manifestly concern the fluctuation field ξ only.

4.3. Generalized Onsager Reciprocity Relations. The following proposition represents a generalization of the Onsager reciprocity relations to nonequilibrium steady states of the nonlinear diffusion process.

Proposition 4.1. *Under the assumption of the regression and local equilibrium hypotheses, the transport coefficients of the nonlinear diffusion process satisfy the position-dependent Onsager relations*

$$K_{kl}\big(\theta(x)\big) = K_{lk}\big(\theta(x)\big) \ \forall \ x\in\Omega, \ \ k, l\in[1, m]. \tag{4.16}$$

Proof. Since we employ the same argument as that for nonequilibrium states of conservative systems in Ref. [15, Ch. 7], we shall just sketch the proof here. We start by introducing the linear transformation $\mathcal{L}_0$ of $\mathcal{D}^m(\Omega)$ by the formula

$$\mathcal{L}_0 := \tilde{K}\big(q(x_0)\big)\Delta. \tag{4.17}$$

It then follows, after some manipulation, from Eqs. (1.10), (3.8), (4.14) and (4.17), together with the continuity properties of the functions $\tilde{K}, \ J$ and q, that

$$\lim_{\epsilon\downarrow 0}\epsilon^2 E\big(\xi([\mathcal{L}^\star - \mathcal{L}_0^\star]f_{x_0,\epsilon})\xi(f'_{x_0,\epsilon})\big) = 0 \ \forall \ f, f'\in\mathcal{D}^m\Omega), \ \ x_0\in\Omega. \tag{4.18}$$

This implies that $\mathcal{L}$ may be replaced by $\mathcal{L}_0$ in Eq. (4.15), i.e. that

$$\lim_{\epsilon\downarrow 0}\epsilon^2 E\big(\xi(\mathcal{L}_0^\star f_{x_0,\epsilon})\xi(f'_{x_0,\epsilon})\big) = \lim_{\epsilon\downarrow 0}\epsilon^2 E\big(\xi(\mathcal{L}_0^\star f'_{x_0,\epsilon})\xi(f_{x_0,\epsilon})\big) \ \forall \ f, f'\in\mathcal{D}^m(\Omega), \ \ x_0\in\Omega. \tag{4.19}$$

Further, since, by Eqs. (4.11) and (4.17),

$$\epsilon^2\mathcal{L}_0 f_{x_0,\epsilon} = [\mathcal{L}_0 f]_{x_0,\epsilon},$$

Eq. (4.19) reduces to the form

$$\lim_{\epsilon\downarrow 0}E\big(\xi([\mathcal{L}_0 f]_{x_0,\epsilon})\xi(f'_{x_0,\epsilon})\big)=\lim_{\epsilon\downarrow 0}E\big(\xi([\mathcal{L}_0 f']_{x_0,\epsilon})\xi(f_{x_0,\epsilon})\big)\ \forall\ f,f'{\in}\mathcal{D}^m(\Omega),\ x_0{\in}\Omega.$$

It follows from this equation, together with Eqs. (3.8), (4.14) and (4.17) that

$$\Big(\Delta f,K\big(\theta(x_0)\big)f'\Big)=\Big(\Delta f',K\big(\theta(x_0)\big)f\Big),\ \forall\ f,f'{\in}\mathcal{D}^m(\Omega),\ x_0{\in}\Omega. \tag{4.20}$$

Further, since, by Eq. (4.8),

$$(\Delta f,f')\equiv(\Delta f',f)\ \forall\ f,f{\in}\mathcal{D}^m(\Omega),$$

and since the actions of Δ and $K\big(\theta(x_0)\big)$ on $\mathcal{D}^m(\Omega)$ intercommute, Eq. (4.20) is equivalent to the following formula.

$$\Big(\Delta f,K\big(\theta(x_0)\big)f'\Big)=\Big(\Delta f,K^\star\big(\theta(x_0)\big)f'\Big)\ \forall\ f,f'{\in}\mathcal{D}^m(\Omega),\ x_0{\in}\Omega, \tag{4.21}$$

where $K^\star$ is the adjoint of K. Hence, the matrix $K\big(\theta(x_0)\big)$ is symmetric for all points x_0 in Ω. This is equivalent to the required result.

5. Fluctuating Currents, Chaoticity and the Onsager- Machlup Process

5.1. A Preliminary Observation. We now aim to extend the stochastic process ξ so as to include the currents associated with these fluctuations. To this end we recall that, under the assumptions of Props. 3.1 and 3.3, $\xi_t^{(N)}$ converges a classical process ξ, indexed by $\mathcal{D}'^m(\Omega)\times\mathbf{R}$, with $\xi_t(f)$ continuous in f and measurable in t. We shall now argue that, by contrast, $\eta^{(N)}$ cannot converge to a process η possessing the corresponding continuity and measurability properties. To show this, we suppose that the correlation functions for $\eta^{(N)}$ converge to those of a process η, indexed by $\mathcal{D}_V'^m(\Omega)\times\mathbf{R}$. Then, since $\mathcal{L}^\star$ is the generator of $T^\star(\mathbf{R}_+)$, it follows from Eqs. (3.15), (3.20), (4.1), (4.5) and (4.6) that

$$\int_0^t ds_1\int_0^t ds_2 E\big(\eta_{s_1}(\nabla f)\eta_{s_2}(\nabla f)\big)=E\big([\xi_t(f)-\xi(f)]^2\big)=$$

$$2E\big(\xi(f)[\xi(f)-\xi(T_t^\star f)]\big)=-2\int_0^t dsW_S(f,T_t^\star\mathcal{L}^\star f))\ \forall f{\in}\mathcal{D}^m(\Omega),\ t{\in}\mathbf{R}_+.$$

Now the r.h.s. of this equation is $O(t)$, whereas the l.h.s. would be $O(t^2)$ if $E\big(\eta_{s_1}(g)\eta_{s_2}(g)\big)$ were continuous in g and measurable with respect to s_1 and s_2. Hence, we cannot assume that $\eta^{(N)}$ converges to a process η that possesses these continuity and measurability properties.

5.2. The Processes ζ and η. In view of this observation, we proceed somewhat differently, starting with the definition

$$\zeta_{t,s}^{(N)}(g):=\int_s^t du\eta_u^{(N)}(g)\ \forall\ g{\in}\mathcal{D}_V^m(\Omega),\ t,s{\in}\mathbf{R}. \tag{5.1}$$

We assume that the cyclic vector $\Phi^{(N)}$ lies in the domain of all monomials in the operators $\xi_u^{(N)}(f)$ and $\zeta_{t,s}^{(N)}(g)$ as f and g run through $\mathcal{D}^m(\Omega)$ and $\mathcal{D}_V^m(\Omega)$, respectively, and t, s and u run through $\mathbf{R}$. We further assume that the correlation functions given by the expectation values of these monomials for the vector state $\Phi^{(N)}$ are continuous in their spatial test functions and time variables, that they converge pointwise to definite limits as $N \to \infty$, and that these limits satisfy the canonical counterparts to the assumptions of Props. (3.1) and (3.3). It then follows, by analogy with the arguments of Section 3, that the quantum process $(\xi^{(N)}, \zeta^{(N)})$ converges to a classical one, (ξ, ζ), whose two components are indexed by $\mathcal{D}^m(\Omega) \times \mathbf{R}$ and $\mathcal{D}_V^m(\Omega) \times \mathbf{R}^2$, respectively, and are continuous with respect to their spatial test functions and measurable with respect to their time variables.

In view of Eq. (5.1) and the fact that the process ζ is the limiting form of $\zeta^{(N)}$ as $N \to \infty$, we term ζ the *time-integrated current*. We note that since by Eq. (5.1),

$$\zeta_{t,s}^{(N)} \equiv \zeta_{t,u}^{(N)} + \zeta_{u,s}^{(N)} \text{ and } \zeta_{t,t}^{(N)} \equiv 0,$$

it follows that, correspondingly,

$$\zeta_{t,s} \equiv \zeta_{t,u} + \zeta_{u,s} \text{ and } \zeta_{t,t} \equiv 0, \tag{5.2}$$

Further, by Eqs. (3.15) and (5.1),

$$\xi_t^{(N)}(f) - \xi_s^{(N)}(f) = \zeta_{t,s}^{(N)}(\nabla f) \ \forall \ f \in \mathcal{D}^m(\Omega),\ t, s \in \mathbf{R},$$

and hence, correspondingly,

$$\xi_t(f) - \xi_s(f) = \zeta_{t,s}(\nabla f) \ \forall \ f \in \mathcal{D}^m(\Omega),\ t, s \in \mathbf{R}, \tag{5.3}$$

which is just the local conservation law for ξ.

5.3. Extension of the Regression Hypothesis: Secular and Stochastic Currents. By Eq. (1.8), the increment δj_t in the phenomenological current due to a perturbation δq_t of the field q_t is $\mathcal{K} \delta q_t$. Correspondingly, by way of extending the regression hypothesis of Section 3, we designate the secular part of the time-integrated fluctuation current $\zeta_{t,s}$ to be

$$\zeta_{t,s}^{sec} := \int_s^t du \mathcal{K} \xi_u, \tag{5.4}$$

where $\mathcal{K}$, defined formally by Eq. (1.8), may now be interpreted as a mapping from $\mathcal{D}'^m(\Omega)$ into $\mathcal{D}_V'^m(\Omega)$. We define the time- integrated stochastic current to be the residual part of $\zeta_{t,s}$, namely

$$\tilde{\zeta}_{t,s} = \zeta_{t,s} - \zeta_{t,s}^{sec},$$

i.e., by Eq. (5.4),

$$\tilde{\zeta}_{t,s} = \zeta_{t,s} - \int_s^t du \mathcal{K} \xi_u. \tag{5.5}$$

In view of this formula, we may re-express the local conservation law (5.3) in the form

$$\xi_t(f) - \xi_s(f) = \int_s^t du \xi_u(\mathcal{K}^\star \nabla f) + \tilde{\zeta}_{t,s}(\nabla f),$$

or equivalently, since Eqs. (1.9) and (3.15) imply that $\nabla.\mathcal{K} = -\mathcal{L}$,

$$\xi_t(f) - \xi_s(f) = \int_s^t du \xi_u(\mathcal{L}^\star f) + w_{t,s}(f) \ \forall \ f \in \mathcal{D}^m(\Omega), \ t, s \in \mathbf{R}, \tag{5.6}$$

where

$$w_{t,s}(f) := \tilde{\zeta}_{t,s}(\nabla f)) \ \forall \ f \in \mathcal{D}^m(\Omega), \ t, s \in \mathbf{R}. \tag{5.7}$$

Further, since, by Eqs. (5.2) and (5.7),

$$w_{t,s} \equiv w_{t,u} + w_{u,s} \text{ and } w_{t,t} \equiv 0, \tag{5.8}$$

Eq. (5.6) is *formally* a Langevin equation. However, the condition for it to qualify as a *bona fide* Langevin equation is that w has the temporal stochastic properties of a Wiener process. The following proposition, which we shall prove in Appendix A, establishes that its two point function does have the requisite properties. Further assumptions concerning the chaoticity of the time-integrated stochastic current $\tilde{\zeta}_t$, which will be introduced in Section 5.4, then lead to a picture in which w is indeed a fully fledged Wiener process.

Proposition 5.1. *Assuming the regression hypothesis, the local conservation law (5.3) and the definition of* w_t,

$$E\big(w_{t,s}(f)\xi_u(f')\big) = 0 \ \forall \ t \geq s \geq u, \ f, f' \in \mathcal{D}^m(\Omega) \tag{5.9}$$

and

$$E\big(w_{t,s}(f)(w_{t',s'}(f')\big) = -\big[W_S(\mathcal{L}^\star f, f') + W_S(f, \mathcal{L}^\star f')\big] |[s,t] \cap [s',t']|$$
$$\forall \ t, s(\leq t), t', s'(\leq t') \in \mathbf{R}, \ f, f' \in \mathcal{D}^m(\Omega), \tag{5.10}$$

where the last factor represents the length of the intersection of the intervals $[s,t]$ *and* $[s',t']$ *and* W_S *is the two- point function defined by Eq. (4.5). Further the process* w *is non- trivial, i.e.* $w_{t,s}$ *does not vanish.*

5.4. The Chaoticity and Temporal Continuity Hypothesss. We assume that the stochastic current is chaotic in the sense that the space-time correlations of $\tilde{\zeta}_{t,s}(x)$ are of short range on the microscopic scale. This assumption is designed to represent Boltzmann's hypothesis of molecular chaos, as transferred from the local particle velocities to the stochastic currents. Since L_N tends to infinity with N, it signifies that the space-time correlations of $\tilde{\zeta}_{t,s}(x)$ are of zero range on the hydrodynamic scale. Further, in accordance with the central limit theorem for fluctuation fields with short range spatial correlations [39], we assume that the process $\tilde{\zeta}$ is Gaussian. Thus, our chaoticity hypothesis is that

(C.1) The process $\tilde{\zeta}$ is Gaussian;

(C.2) $E\big(\tilde{\zeta}_{t,s}(g)\tilde{\zeta}_{t',s'}(g')\big)=0$ if $(s,t)\cap(s',t')=\emptyset$; and

(C.3) $E\big(\tilde{\zeta}_{t,s}(g)\tilde{\zeta}_{t',s'}(g')\big)=0$ if $\mathrm{supp}(g)\cap\mathrm{supp}(g')=\emptyset$.

It follows immediately from **(C.1)** that the process $\tilde{\zeta}$ is completely determined by its two-point function $E\big(\tilde{\zeta}_{t,s}(g)\tilde{\zeta}_{t',s'}(g')\big)$. In view of the discussion following Eq. (5.1), this is continuous with respect to the test functions g and g' and measurable with respect to the time variables t, s, t' and s'. We now strengthen this conclusion by the following continuity hypothesis to the effect that it is continuous with respect to the time variables.

($\mathcal{C}$) The two-point function $E\big(\tilde{\zeta}_{t,s}(g)\tilde{\zeta}_{t',s'}(g')\big)$ is continuous with respect to the time variables t, s, t', s'.

The following proposition, which we shall prove in Appendix B, stems from a application of a key theorem of Schwartz [33, Theorem 35] to the process $\tilde{\zeta}$, subject to the assumptions **(C.2)** and ($\mathcal{C}$).

Proposition 5.2. *Under the assumption of the hypotheses* **(C.2)**, **(C.3)** *and* ($\mathcal{C}$), *together with the condition of continuity with respect to its spatial test functions, the two-point function for the process* $\tilde{\zeta}$ *takes the form*

$$E\big(\tilde{\zeta}_{t,s}(g)\tilde{\zeta}_{t',s'}(g')\big)=\Gamma(g,g')|[s,t]\cap[s',t']|\ \forall\ g,g'\in\mathcal{D}_V^m(\Omega),\ \ t,s,t',s'\in\mathbf{R},\tag{5.11}$$

where $\Gamma\in\mathcal{D}_V'^m(\Omega)\otimes\mathcal{D}_V'^m(\Omega)$ *and* $\mathrm{supp}\Gamma\subset\{(x,x')\in\Omega^2|x'=x\}$.

5.5. A Local Equilibrium Condition for the Currents.

In order to extend our local equilibrium condition to the stochastic currents of the nonlinear diffusion process, we start by formulating the two point function at equilibrium for the process $\tilde{\zeta}$.

Equilibrium Two Point Function for $\tilde{\zeta}$. Assuming again that the field q is uniform at equilibrium, we infer from Eqs. (1.6) and (1.7) that in this situation $\mathcal{L}=\tilde{K}(q)\Delta$, with q constant. Hence, by Eqs. (3.8), (4.7), (5.7) and (5.10), together with the symmetry of $J(q)$, which follows from Eq. (2.6),

$$E_{eq}\big(\tilde{\zeta}_{t,s}(\nabla f)\tilde{\zeta}_{t',s'}(\nabla f')\big)=-\Big[\big(\Delta f,K(\theta)f'\big)+\big(K(\theta)f,\Delta f'\big)\Big]|[s,t]\cap[s',t']|,$$

which, by Eq. (4.16), is equivalent to the following formula for the unsmeared two-point function for $\tilde{\zeta}$.

$$\frac{\partial^2}{\partial x_\mu\partial x'_\nu}E_{eq}\big(\tilde{\zeta}_{t,s;k,\mu}(x)\tilde{\zeta}_{t',s';l\nu}(x')\big)=-2K_{kl}(\theta)\Delta\delta(x-x')|[s,t]\cap[s',t']|,\tag{5.12}$$

where $\tilde{\zeta}_{t,s;k,\mu}$ is the μ'th spatial component of the k'th component of the field $\tilde{\zeta}_{t,s}=(\tilde{\zeta}_{t,s;1},..,\tilde{\zeta}_{t,s;m})$ and the summation convention is employed for the indices μ and ν. Recalling now our assumption, at the start of Section 2, that the interactions are translationally and rotationally invariant, we assume that the corresponding symmetries are unbroken

in the pure equilibrium phase and thus that the process $\tilde{\zeta}$ is invariant under the space translations and rotations that are implemented within the confines of Ω. We remark here that the limitation in Euclidean symmetry imposed by the boundedness of Ω is not serious from the physical standpoint, since Ω is an open subset of X and so any point of it, as viewed in the microscopic picture, is infinitely far from the boundary of Σ.

Assuming then that the equilibrium two-point function for $\tilde{\zeta}$ is invariant under space translations and rotations, we may express it in the form

$$E_{eq}\big(\tilde{\zeta}_{t,s;k,\mu}(x)\tilde{\zeta}_{t',s';l,\nu}(x')\big) = S_{kl}(x-x')\delta_{\mu\nu}|[s,t]\cap[s',t']|, \tag{5.13}$$

where $S_{kl}\in\mathcal{D}'(\Omega)$. It follows from this formula that Eq. (5.12) reduces to the following differential equation for S_{kl}.

$$\Delta S_{kl}(x) = 2K_{kl}(\theta)\Delta\delta(x). \tag{5.14}$$

Further, by condition **(C.3)** and Eq. (5.13), the distribution S_{kl} has support at the origin, and therefore [33, Theorem 35] $S_{kl}(x-x',t)$ is a finite linear combination of $\delta(x-x')$ and its derivatives. Hence the only admissible solution of Eq. (5.14) is

$$S_{kl}(x) = 2K_{kl}(\theta)\delta(x)$$

and therefore, by Eq. (5.13), the equilibrium two-point function for $\tilde{\zeta}$ is given by the formula

$$E_{eq}\big(\tilde{\zeta}_{t,s;k,\mu}(x)\tilde{\zeta}_{t',s';l,\nu}(x')\big) = 2K_{kl}(\theta)\delta(x-x')\delta_{\mu\nu}|[s,t]\cap[s',t']|\ . \tag{5.15}$$

Equivalently, the equilibrium two-point function for the smeared field $\tilde{\zeta}_{t,s}(g)$ takes the form

$$E_{eq}\big(\tilde{\zeta}_{t,s}(g)\tilde{\zeta}_{t',s'}(g')\big) = 2\big(g,K(\theta)g'\big)_V|[s,t]\cap[s',t']|$$

$$\forall\ g,g'\in\mathcal{D}_V^m(\Omega),\ t,s,t',s'\in\mathbf{R}, \tag{5.16}$$

where $(.)_V$ is the inner product in $\mathcal{D}_V^m(\Omega)$ defined by the formula

$$(g,g')_V = \sum\nolimits_{k=1}^m \int_\Omega dx g(x).g'(x)\ \forall\ g,g'\in\mathcal{D}_V^m(\Omega)\equiv$$

$$\sum\nolimits_{k=1}^m\sum\nolimits_{\mu=1}^d \int_\Omega dx g_{k,\mu}(x)g'_{k,\mu}(x), \tag{5.17}$$

and where $g_{k,\mu}$ is the μ'th spatial component of g_k.

Local Property of the Equilibrium Two Point Function. We formulate the local properties of the stochastic current $\tilde{\zeta}$ along the lines employed in Section 4.2 for the process ξ. Thus, for $(x_0,\epsilon)\in\Omega\times\mathbf{R}_+$, and ϵ sufficiently small, we define the transformation $g\to g_{x_0,\epsilon}$ of $\mathcal{D}_V^m(\Omega)$ by the formula

$$g_{x_0,\epsilon}(x) = \epsilon^{-d/2}g\big(\epsilon^{-1}(x-x_0)\big). \tag{5.18}$$

We then observe that, by Eqs. (5.17) and (5.18), the transformations $t \to \epsilon^2 t,\ g \to g_{x_0,\epsilon}$, of the times and test functions lead to the multiplication of the smeared two-point function of Eq. (5.16) by the factor ϵ^2. Thus,

$$\epsilon^{-2} E_{eq}\big(\tilde{\zeta}_{\epsilon^2 t,\epsilon^2 s}(g_{x_0,\epsilon})\tilde{\zeta}(g'_{x_0,\epsilon})\big) = 2\big(g, K(\theta)g'\big)_V |[s,t]\cap[s',t']|$$

$$\forall\ x_0 \in \Omega,\ \ g, g' \in \mathcal{D}_V^m(\Omega),\ \ t, s, t', s' \in \mathbf{R}. \tag{5.19}$$

The local property of the two-point function for $\tilde{\zeta}$ at the point x_0 is then obtained by passing to the limiting form of this equation as $\epsilon \to 0$.

Local Equilibrium Property for the Stochastic Current in the Nonequilibrium Steady State. In view of the last observation, we assume that, in the nonequilibrium steady state, the process $\tilde{\zeta}$ enjoys the local equilibrium property obtained by passing to the limit $\epsilon \to 0$ and replacing E_{eq} and θ by E and $\theta(x_0)$, respectively, in Eq. (5.19). Thus we assume that

$$\lim_{\epsilon \to 0} \epsilon^{-2} E\big(\tilde{\zeta}_{\epsilon^2 t,\epsilon^2 s}(g_{x_0,\epsilon})\tilde{\zeta}_{\epsilon^2 t',\epsilon^2 s'}(g'_{x_0,\epsilon})\big) = 2\big(g, K(\theta(x_0))g'\big)_V |[s,t]\cap[s',t']|$$

$$\forall\ x_0 \in \Omega,\ \ g, g' \in \mathcal{D}_V^m(\Omega),\ \ t, s(\le t), t', s'(\le t') \in \mathbf{R}. \tag{5.20}$$

This is our local equilibrium condition for the stochastic current.

5.6. Explicit Form of the Two Point Function for $\tilde{\zeta}$. By Prop. 5.2, this function is determined by the functional Γ, which by Eqs. (5.11) and (5.20), possesses the following local equilibrium property.

$$\lim_{\epsilon \downarrow 0} \Gamma(g_{x_0,\epsilon}, g'_{x_0,\epsilon}) = 2\big(g, K(\theta(x_0))g'\big)_V \forall\ x_0 \in \Omega,\ \ g, g' \in \mathcal{D}_V^m(\Omega). \tag{5.21}$$

The following proposition, which will be proved in Appendix C, provides an explicit formula for the functional Γ, which stems from a combination of the chaoticity condition **(C.3)** and the local equilibrium condition (5.21).

Proposition 5.3. *Under the previous assumptions, together with the local equilibrium condition of (5.21),* Γ *is given by the formula*

$$\Gamma(g, g') = 2(g, K_\theta g')_V\ \ \forall\ g, g' \in \mathcal{D}_V^m(\Omega), \tag{5.22}$$

where K_θ *is the matrix-valued operator* $K \circ \theta$ *in* $\mathcal{D}_V^m(\Omega)$*, i.e.*

$$K_\theta(x) = K\big(\theta(x)\big). \tag{5.23}$$

The following corollary is an immediate consequence of this proposition and Prop. 5.2.

Corollary 5.4. *Under the same assumptions, the two-point function of the stationary process* $\tilde{\zeta}$ *is given by the formula*

$$E\big(\tilde{\zeta}_{t,s}(g)\tilde{\zeta}_{t',s'}(g')\big) = 2(g, K_\theta g')_V |[s,t]\cap[s',t']|$$

$$\forall\ g, g' \in \mathcal{D}_V^m(\Omega),\ t, s(\leq t), t', s'(\leq t') \in \mathbf{R}. \tag{5.24}$$

5.7. The Generalized Onsager-Machlup Process ξ.

It now follows immediately from Cor. 5.4 and Eq. (5.7) that

$$E\big(w_{t,s}(f) w_{t',s'}(f')\big) = 2(\nabla f, K_\theta \nabla f')_V |[s,t] \cap [s',t']|$$

$$\forall\ f, f' \in \mathcal{D}^m(\Omega),\ t, s, t', s' \in \mathbf{R}. \tag{5.25}$$

Hence, by the chaotic hypothesis **(C.1)** and Eqs. (5.7) and (5.25), w is a generalized Wiener process. Further, on re-expressing Eq. (5.6) in the form

$$d\xi_t = \mathcal{L}\xi_t dt + dw_{t,s}, \tag{5.26}$$

we see that, in view of the additive property (5.8) of w, the fluctuation field ξ executes a it generalized Onsager- Machlup process; while Eq. (5.25) signifies that the two-point function for w corresponds precisely to that assumed for the stochastic force in Landau's fluctuation hydrodynamics [18].

In order to derive the properties of the process ξ from those of w, we note that, since $\mathcal{L}$ is the generator of $T(\mathbf{R}_+)$, the solution of the Langevin equation (5.26) is given by the formula

$$\xi_t = T_{t-s}\xi_s + \int_s^t T_{t-u} dw_{u,s}\ \forall\ t, s(\leq t) \in \mathbf{R}, \tag{5.27}$$

or equivalently,

$$\xi_t(f) = \xi_s(T^\star_{t-s} f) + \int_s^t dw_{u,s}(T^\star_{t-u} f)\ \forall\ f \in \mathcal{D}^m(\Omega),\ T, s(\leq t) \in \mathbf{R}_+. \tag{5.28}$$

The following proposition, which we shall prove in Appendix D, is a natural generalization of standard properties of the Brownian motion of a single particle that ensue from the Langevin equation governing its velocity (cf. [36]).

Proposition 5.5. *Under the above assumptions,*

(1) ξ is a Gaussian Markov process, and

(2) the fields $w_{t,s}$ and ξ_u are statistically independent of one another if s and t are greater than or equal to u.

Comment. It follows from this proposition and Eqs. (4.5) and (4.6) that the process ξ is completely determined by the forms of the semigroup $T^\star(\mathbf{R}_+)$ and the distribution W_S.

6. Long Range Spatial Correlations of the ξ- Process

6.1. The Static Two-Point Function for ξ. By Eq. (4.5), the unsmeared form of the $\mathcal{D}'^m(\Omega)\otimes\mathcal{D}'^m(\Omega)$-class distribution W_S is given by the formula

$$W_S(x,x') = E\big(\xi(x)\otimes\xi(x')\big). \tag{6.1}$$

The following Proposition provides an explicit formula for W_S, as well as a differential equation for this distribution in terms of the semigroup $T^\star(\mathbf{R}_+)$, and the transport function K_θ.

Proposition 6.1. *Under the above assumptions,*

$$W_S(f,f') = 2\int_0^\infty dt\big(\nabla T_t^\star f, K_\theta\nabla T_t^\star f'\big)_V \ \forall f,f'\in\mathcal{D}^m(\Omega) \tag{6.2}$$

and, further, the generalized function $W_S(x,x')$ satisfies the equation

$$[\mathcal{L}\otimes I + I\otimes\mathcal{L}']W_S(x,x') = 2\nabla.\big(K_\theta(x)\nabla\delta(x-x')\big), \tag{6.3}$$

where $\mathcal{L}'$ is the version of $\mathcal{L}$ that acts on functions of x'.

Proof. By Eq. (4.5) and the stationarity of the ξ-process,

$$W_S(f,f') = E\big(\xi_t(f)\xi_t(f')\big) \ \forall\ f,f'\in\mathcal{D}^m(\Omega),\ t\in\mathbf{R}_+$$

and therefore, by Eq. (5.26),

$$W_S(f,f') = E\big(\xi(T_t^\star f)\xi(T_t^\star f')\big) + \int_0^t E\big(\xi(T_t^\star f)dw_{u,0}(T_{t-u}^\star f')\big) +$$

$$\int_0^t E\big(\xi(T_t^\star f')dw_{u,0}(T_{t-u}^\star f)\big) + \int_0^t\int_0^t E\big(dw_{u,0}(T_{t-u}^\star f)dw_{u',0}(T_{t-u'}^\star f')\big)$$
$$\forall\ f,f'\in\mathcal{D}^m(\Omega),\ t\in\mathbf{R}_+. \tag{6.4}$$

Now, by the dissipativity condition (3.12), the first term on the r.h.s. of this equation vanishes in the limit $t\to\infty$, while by Eq. (5.9), the second and third terms there vanish. Hence, it follows from Eq. (6.4) that

$$W_S(f,f') = \lim_{t\to\infty}\int_0^t\int_0^t E\big(dw_{u,0}(T_{t-u}^\star f)dw_{u',0}(T_{t-u'}^\star f')\big) \ \forall\ f,f'\in\mathcal{D}^m(\Omega). \tag{6.5}$$

Further, by Eq. (5.25),

$$E\big(dw_{u,0}(f)dw_{u',0}(f')\big) = 2(\nabla f, K_\theta\nabla f')_V\delta(u-u')dudu'$$

and consequently Eq. (6.5) reduces to the form

$$W_S(f,f') = \lim_{t\to\infty}2\int_0^t du(\nabla T_{t-u}^\star f, \nabla T_{t-u}^\star f')_V \equiv 2\int_0^t du(\nabla T_u^\star f, \nabla T_u^\star f')_V,$$

which is equivalent to the required formula (6.2).

Further, since $\mathcal{L}^\star$ is the generator of $T^\star(\mathbf{R}_+)$, it follows from Eq. (6.2) that

$$W_S(\mathcal{L}^\star f, f') + W_S(f, \mathcal{L}^\star f') = 2\int_0^\infty dt \frac{d}{dt}(\nabla T_t^\star f, K_\theta \nabla T_t^\star f')_V$$

and consequently, by the dissipativity condition (3.12),

$$W_S(\mathcal{L}^\star f, f') + W_S(f, \mathcal{L}^\star f') = -2(\nabla f, K_\theta \nabla f')_V \ \forall \ f, f' \in \mathcal{D}^m(\Omega),$$

which, by Eq. (6.1), is equivalent to the required formula (6.3).

6.2. Long Range Spatial Correlations. In order to provide a precise characterization of long range correlations, we first recall that the ratio of the macroscopic length scale to the microscopic one is infinite. Consequently, correlations of finite range on the microscopic scale are of zero range on the macroscopic one. Accordingly, we term the range of correlations 'short' or 'long' according to whether or not it reduces to zero in the macroscopic picture. Thus our condition for long range spatial correlations for the ξ-field is simply that the support of the distribution W_S does *not* lie in the domain $\{(x, x') \in \Omega^2 | x = x'\}$. The following proposition establishes that the spatial correlations of ξ for the nonlinear diffusion process are generically of long range.

Proposition 6.2. *Let Φ_q be the m-by-m matrix-valued function on Ω defined by the formula*

$$\Phi_q(x) = \Delta K_\theta(x) + \nabla.\Psi_q(x), \tag{6.6}$$

where

$$\Psi_{q;kl}(q;x) = \sum\nolimits_{k',l'=1}^m \Big[\frac{\partial}{\partial q_{l'}(x)} \tilde{K}_{kk'}\big(q(x)\big)\Big] \big[J_{l'l}\big(q(x)\big)\nabla q_{k'}(x) - J_{k'l}\big(q(x)\big)\nabla q_{l'}(x)\big]. \tag{6.7}$$

Then under the above assumptions, a sufficient condition for the spatial correlations of ξ to be of long range is that either Φ_q does not vanish or that the matrix Ψ_q is symmetric.

Comments. (1) The Proposition establishes that the correlations are generically of long range, since the specified conditions on Φ_q and Ψ_q can be satified only for special relationships between the functions $\tilde{K} \circ q$ and $s \circ q$; and these are generally independent of one another, since s and $\tilde{K}$ govern the equilibrium and transport properties, respectively, of Σ. By contrast, the corresponding correlations for equilibrium states are generically of short range, except at critical points. A treatment of critical equilibrium correlations of fluctuation observables is provided by Ref. [40].

(2) In the particular case of the symmetric exclusion process [9-11], $n = 1$, $d = 1$, $\tilde{K}(q) = 1$, $s(q) = -q\ln q - (1-q)\ln(1-q)$ and $q(x) = a + b.x$, where a and b ($\neq 0$) are constants. Thus, in this case, it follows from Eqs. (1.6), (2.6), (6.6) and (6.7) that $\Psi_q = 0$ and $\Phi_q(x) = -2b^2 \neq 0$. Hence, long range correlations prevail in this model, in accordance with the results obtained by its explicit solution in Refs. [9-11].

Proof of Prop. 6.2. Suppose that the static spatial correlations of ξ are not of long range, i.e. that the support of the distribution W_S lies in the domain $\{(x,x')\in\Omega^2 \vert x'=x\}$. Then it follows from this supposition and the local equilibrium condition (4.14), by precise analogy of the derivation of Eq. (5.24) from corresponding conditions of zero range correlations and local equilibrium for the process $\tilde{\zeta}$, that

$$W_S(x,x') = J_q(x)\delta(x-x'), \tag{6.8}$$

where

$$J_q(x) := J\bigl(q(x)\bigr). \tag{6.9}$$

Hence, by Eqs. (1.10), (3.8) and (6.7)-(6.9),

$$(\mathcal{L}\otimes I)W(x,x') = \Delta[K_\theta(x)\delta(x-x')] + \nabla.[\Psi_q(x)\delta(x-x')]. \tag{6.10}$$

Further, by Eq. (6.1),

$$(I\otimes\mathcal{L}')W(x,x') = [(\mathcal{L}'\otimes I)W(x',x)]^{tr},$$

where the superscript tr denotes transpose, and therefore, by Eq. (6.10),

$$(I\otimes\mathcal{L}')W(x,x') = \Delta'[K_\theta(x'-x)\delta(x'-x)]^{tr} + \nabla'.[\Psi_q(x')\delta(x'-x)]^{tr}, \tag{6.11}$$

where Δ' and ∇' are the versions of Δ and ∇, respectively, that act on functions of x'. Consequently, since K_θ is symmetric, by Eqs. (4.16) and (5.23), it follows from Eqs. (6.6), (6.10) and (6.11) that

$$[\mathcal{L}\otimes I + I\otimes\mathcal{L}']W_S(x,x') =$$

$$2\nabla.\bigl(K_\theta(x)\nabla\delta(x-x')\bigr) + \Phi_q(x)\delta(x-x') + [\Psi_q(x) - \Psi_q^{tr}(x)].\nabla\delta(x-x'). \tag{6.12}$$

On comparing this equation with Eq. (6.3), we see that

$$\Phi_q(x)\delta(x-x') + [\Psi_q(x) - \Psi_q^{tr}(x)].\nabla\delta(x-x') = 0,$$

i.e. that Φ_q vanishes and that Ψ_q is symmetric. These, then, are conditions that ensue from the assumption of short range correlations of the ξ-process. We conclude, therefore, that the violation of either of these conditions signifies that the correlations are of long range.

7. Concluding Remarks.

We have proposed a macrostatistical treatment of nonequilibrium stady states of quantum systems that is centred on the fluctuations of their hydrodynamical variables. The key physical assumptions on which this treatment is based are

(a) the regression hypothesis for the hydrodynamic fluctuation field ξ;

(b) the chaoticity of the associated currents, as represented by their time integrals $\zeta_{t,s}$;

(c) the local equilibrium conditions on the stochastic process comprising ξ and ζ;

(d) the space-time scale invariance of the phenomenological equation of motion (1.4), as exemplified by the case of nonlinear diffusions; and

(e) the invariance of the quantum field $\hat{q}$, and correspondingly of the classical field ξ, under time reversals.

On the basis of these assumptions and certain technical ones, we have obtained a picture that provides natural generalizations of the Onsager reciprocity relations and the Onsager-Machlup fluctuation process to nonequilibrium steady states, together with a demonstration that the spatial correlations of the hydrodynamical variables are generically of long range in these states. Furthermore this picture is expressed exclusively in terms of the phenomenological functions representing the equilibrium entropy, $s(q)$, the transport coefficients $K(\theta)$ and the hydrodynamical boundary conditions. This may easily be seen from the comment at the end of Section 5, together with Eqs. (1.10), (3.8) and (6.2) and the fact that the semigroup $T(\mathbf{R})$ is completely determined by its generator $\mathcal{L}$.

Let us now discuss the assumptions (a)-(e) a little further. In our view, for reasons expressed in Sections 4.1, 4.2 and 5.4, the first three of these seem natural from the physical standpoint, though they are very hard to prove in concrete cases. On the other hand, it is clear that assumptions (d) and (e) are not universally valid: for example, they both fail in the important case of Navier-Stokes hydrodynamics. Consequently, it is of interest to consider how the macrostatistical picture presented here might be extended to situations where (d) and (e) are replaced by weaker assumptions.

In fact, the weakening of (e) provides no serious problems, since the locally conserved fields of continuum mechanics are generally either even or odd with respect to time reversals [41]. Accordingly, we replace (e) by the assumption that each of the quantum fields $\hat{q}_j$ has either even or odd parity with respect to time reversals, i.e. that

$$\tau\hat{q}_j(x) = R_j\hat{q}_j(x),\ R_j = \pm 1,\ j = 1, . ., n, \tag{7.1}$$

where again τ is the time-reversal antiautomorphism. This weakened assumption then leads to the nonlinear version of Casimir's extension [41] of Onsager's theory, wherein Eq. (4.16) is modified to the formula

$$K_{kl}\big(\theta(x_0)\big) = R_k R_l K_{lk}\big(R\theta(x_0)\big), \tag{7.2}$$

where

$$R(\theta) := \big(R_1\theta_1, . ., R_n\theta_n\big). \tag{7.3}$$

Similarly, the modification of assumption (e) to the form given by Eq. (7.1) presents no serious problems for the other issues treated here.

On the other hand, there does not appear to be any natural generalisation of the scaling assumption (d), which lay behind the interdependence of the ratios of the macroscopic to microscopic scales for distance and time, the former ratio being L_N and the latter L_N^2

(or more generally L_N^k). Moreover, one sees from Eqs. (2.15) and (3.13) that this interdependence was essential to the limit procedures of Eqs. (3.1) and (3.20). Nevertheless it does not appear to be essential to the key physical ideas that

(i) the ratios of the macroscopic to microscopic scales for both distance and time are extremely large, and

(ii) the currents associated with the locally conserved quantum fields satisfy the chaoticity assumption of Section 5.4, whereby the space-time correlations of their fluctuations decay within microscopic distances and times.

Since such chaoticity does not necessarily require any interdependence of the ratios of the macroscopic to microscopic scales for distance and time, it appears reasonable to expect that some version of the present macrostatistical model should still be applicable even in the absence of macroscopic space-time scale invariance.

Thus, from the standpoint of mathematical physics, a most challenging question is whether the present scheme can be generalized to a setting which does not require the scale invariance of the macroscopic law (1.4). Presumably such a generalization would require a difficult multi-scale analysis.

Appendix A: Proof of Proposition 5.1

We shall first prove Eqs. (5.9) and (5.10) and then demonstrate the nontriviality of the process w.

Since $\mathcal{L}$ is the generator of $T(\mathbf{R}_+)$, Eq. (5.9) follows immediately from Eqs. (4.2) and (5.6).

It then follows from Eqs. (5.6) and (5.9) that the l.h.s. of Eq. (5.10) vanishes if the intervals $[s,t]$ and $[s',t']$ do not intersect. Hence, in view of Eq. (5.8), the proof of Eq. (5.10) reduces to that of the same formula with $s=s'$ and $t=t'$ and $t\geq s$. Thus it suffices for us to prove that

$$E\big(w_{t,s}(f)w_{t,s}(f')\big) = -\big(W_S(\mathcal{L}^\star f,f') + W_S(f,\mathcal{L}^\star f')\big)|t-s|$$

$$\forall\ t,s\ (\leq t)\in\mathbf{R},\ \ f,f'\in\mathcal{D}^m(\Omega). \tag{A.1}$$

We start by inferring from Eq. (5.6) that the l.h.s. of Eq. (A.1) is the sum of the following four terms:-

$$E\big[\big(\xi_t(f)-\xi_s(f)\big)\big(\xi_t(f')-\xi_s(f')\big)\big], \tag{a}$$

$$-\int_s^t du E\big[\big(\xi_t(f)-\xi_s(f)\big)\xi_u(\mathcal{L}^\star f')\big], \tag{b}$$

$$-\int_s^t du E\big[\xi_u(\mathcal{L}^\star f)\big(\xi_t(f')-\xi_s(f')\big)\big] \tag{c}$$

and

$$\int_s^t du \int_s^t dv E\big(\xi_u(\mathcal{L}^\star f)\xi_v(\mathcal{L}^\star f)\big). \tag{d}$$

Since $t\geq s$ and the ξ-process is stationary, it follows from Eqs. (4.5) and (4.6) that

$$\text{Term } (a) = 2W_S(f,f') - W_S(T^\star_{t-s}f,f') - W_S(f,T^\star_{t-s}f'), \tag{$A.2$}$$

$$\text{Term } (b) = -\int_s^t du W_S(T^\star_{t-u}f,\mathcal{L}^\star f') + \int_s^t du W_S(f,T^\star_{u-s}\mathcal{L}^\star f'), \tag{$A.3$}$$

$$\text{Term } (c) = -\int_s^t du W_S(\mathcal{L}^\star f,T^\star_{t-u}f') + \int_s^t du W_S(T^\star_{u-s}\mathcal{L}^\star f,f') \tag{$A.4$}$$

and

$$\text{Term } (d) = \int_s^t du \int_s^u dv W_S(T^\star_{u-v}\mathcal{L}^\star f,\mathcal{L}^\star f') + \int_s^t du \int_u^t dv W_S(\mathcal{L}^\star f,T^\star_{v-u}\mathcal{L}^\star f'). \tag{$A.5$}$$

Since W_S is linear in each of its arguments and since $\mathcal{L}^\star$ is the generator of $T^\star(\mathbf{R}_+)$, it follows that Eqs. (A.3-5) may be re-expressed in the following forms.

$$\text{Term } (b) = -\int_s^t du W_S(T^\star_{t-u}f,\mathcal{L}^\star f') + W_S(f,T^\star_{t-s}f') - W_S(f,f'), \tag{$A.6$}$$

$$\text{Term } (c) = -\int_s^t du W_S(\mathcal{L}^\star f,T^\star_{t-u}f') + W_S(T^\star_{t-s}f,f') - W_S(f,f') \tag{$A.7$}$$

and

$$\text{Term } (d) = \int_s^t du\big[-W_S(f,\mathcal{L}^\star f') + W_S(T^\star_{u-s}f,\mathcal{L}^\star f') + W_S(\mathcal{L}^\star f,T^\star_{t-u}f') - W_S(\mathcal{L}^\star f,f')\big]. \tag{$A.8$}$$

It follows now from Eqs. (A.2) and (A.6-8) that the sum of the terms (a), (b), (c) and (d), which comprises the l.h.s. of Eq. (A.1), is equal to the r.h.s. of that equation. This completes the proof of Eq. (A.1) and thus of Eq. (5.10).

Finally, we employ a *reductio ad absurdum* method to establish the nontriviality of the process w. Thus, we assume that $w_{t,s}$ vanishes. It then follows from Eq. (5.7) that

$$W_S(\mathcal{L}^\star f,f') + W_S(f,\mathcal{L}^\star f') = 0 \ \forall \ f,f'\in\mathcal{D}^m(\Omega)$$

and hence that

$$W_S(\mathcal{L}^\star T^\star_t f,T^\star_t f') + W_S(T^\star_t f,\mathcal{L}^\star T^\star_t f') = 0 \ \forall \ f,f'\in\mathcal{D}^m(\Omega),\ t\in\mathbf{R}_+.$$

Since W_S is linear in each of its arguments and since $\mathcal{L}^\star$ is the generator of $T^\star(\mathbf{R}_+)$, this signifies that

$$\frac{d}{dt}W_S(T_t^\star f, T_t^\star f') = 0$$

and therefore, since $T_0 = I$, that

$$W_S(T_t^\star f, T_t^\star f') = W_S(f, f') \ \forall \ f, f'\in\mathcal{D}^m(\Omega), \ t\in\mathbf{R}_+. \qquad (A.9)$$

Moreover, by Eq. (4.5) and the dissipativity condition (3.12), the l.h.s. of Eq. (A.9) vanishes in the limit $t\to\infty$. Hence Eq. (A.9) implies that the static two-point function W_S vanishes. This conflicts with the fact that, by Eqs. (4.5), (4.11) and (4.14),

$$\lim_{\epsilon\downarrow 0}W_S(f_{x_0,\epsilon}, f'_{x_0,\epsilon}) = \big(f, J(q(x_0))f'\big),$$

which does not vanish identically. This contradiction establishes that the assumption of the triviality of w is untenable and thus completes the proof of the proposition.

Appendix B: Proof of Proposition 5.2.

We start by noting that, in view of Eq. (5.2) and condition **(C.2)**, the proof of this proposition reduces to that of the formula (5.11) for the particular case where $s = s', \ t = t'$ and $s\leq t$. Thus we need only prove that

$$E\big(\zeta_{t,s}(g)\zeta_{t,s}(g')\big) = \Gamma(g, g')(t-s) \ \forall \ g, g'\in\mathcal{D}_V^m(\Omega), \ t, s(\leq t)\in\mathbf{R}, \qquad (B.1)$$

where Γ is an element of $\mathcal{D}_V'^m\otimes\mathcal{D}_V'^m$ with support in the domain $\{(x, x')\in\Omega^2 | x' = x\}$.

To this end, we start by defining

$$F_{g,g'}(t, s) := E\big(\zeta_{t,s}(g)\zeta_{t,s}(g')\big) \qquad (B.2)$$

and inferring from Eq. (5.2) and condition **(C.2)** that

$$F_{g,g'}(t, s) = F_{g,g'}(t, u) + F_{g,g'}(u, s) \text{ for } t\geq u\geq s. \qquad (B.3)$$

Further, by Eq. (B.2) and the stationarity of the process ζ,

$$F_{g,g'}(t, s) = F_{g,g'}(t+b, s+b) \ \forall \ b\in\mathbf{R},$$

which signifies that $F_{g,g'}$ may be expressed in the form

$$F_{g,g'}(t, s) = \tilde{F}_{g,g'}(t-s) \ \forall \ s, t\in\mathbf{R}, \qquad (B.4)$$

where, by condition $(\mathcal{C})$, $\tilde{F}_{g,g'}$ is a continuous function on $\mathbf{R}$. It follows now from Eqs. (B.3) and (B.4) that

$$\tilde{F}_{g,g'}(t) + \tilde{F}_{g,g'}(t') = \tilde{F}_{g,g'}(t+t') \ \forall \ t, t'\in\mathbf{R}_+ \qquad (B.5)$$

and hence that

$$\tilde{F}_{g,g'}(nt) = n\tilde{F}_{g,g'}(t),\ \forall\ t{\in}\mathbf{R}_+,\ n{\in}\mathbf{N}$$

or equivalently

$$\tilde{F}_{g,g'}(t) = n'\tilde{F}_{g,g'}(t/n'),\ \forall\ t{\in}\mathbf{R}_+,\ n'{\in}\mathbf{N}\backslash\{0\}.$$

These last two equations imply that

$$\tilde{F}_{g,g'}(rt) = r\tilde{F}_{g,g'}(t)$$

for all non-negative t and positive rational r; and further, by condition $(\mathcal{C})$, this result extends to all positive r. Hence the action of $\tilde{F}_{g,g'}$ on $\mathbf{R}_+$ takes the form

$$\tilde{F}_{g,g'}(t) = \Gamma(g,g')t\ \forall\ t{\in}\mathbf{R}_+, \tag{B.6}$$

where $\Gamma(g,g') := \tilde{F}_{g,g'}(1)$. By Eqs. (B.2) and (B.4), Eq. (B.6) is equivalent to the required formula (B.1); and, moreover, it follows from condition **(C.2)** and the continuity and linearity of the l.h.s. of that equation with respect to the test functions g and g' that Γ is indeed an element of $\mathcal{D}_V'^m{\otimes}\mathcal{D}_V'^m$ with support in the domain $\{(x,x'){\in}\Omega^2|x'=x\}$.

Appendix C: Proof of Proposition 5.3.

We base the proof of Prop. 5.2 on the following lemma.

Lemma C.1 *Let Ω_1 be any open subset of Ω whose boundary, $\partial\Omega_1$, does not intersect $\partial\Omega$. Then, under the assumptions of Prop. 5.2, the restriction of the two-point function Γ to the spatial domain Ω_1^2 is given by a finite sum of the following form.*

$$\Gamma(g,g') = \sum\nolimits_{n,n'\in\mathbf{N}^d}\sum\nolimits_{k,l=1}^{m}\sum\nolimits_{\mu,\nu=1}^{d}\int_\Omega dx C_{k,l;\mu,\nu}^{n,n'}(x)\partial_x^n g_{k,\mu}(x)\partial_x^{n'}g'_{l,\nu}(x)$$

$$\forall\ g,g'{\in}\mathcal{D}_V^m(\Omega_1),\ t,s,t',s'{\in}\mathbf{R}, \tag{C.1}$$

where

(i) the C's are continuous functions on Ω with support in some arbitrary neighbourhood of Ω_1;

(ii) $g_{k,\mu}$ is the μ'th spatial component of the k'th component of $g=(g_1,.\ ,g_m)$; and

(iii) for $n=(n_1,.\ .,n_d){\in}\mathbf{N}^d$, $\partial_x^n := \partial^{n_1+\ .+n_d}/\partial x_1^{n_1}..\ \partial x_d^{n_d}$.

Proof of Prop. 5.3 assuming Lemma C.1. We start by inferring from Eq. (5.18) that, for any $g,g'{\in}\mathcal{D}_V^m(\Omega)$, $x_0{\in}\Omega$ and ϵ sufficiently small, one can find an open subset Ω_1 of Ω such that $g_{x_0,\epsilon}$ and $g'_{x_0,\epsilon}$ lie in $\mathcal{D}_V^m(\Omega_1)$. Hence, by Eqs. (5.18) and (C.1),

$$\Gamma(g_{x_0,\epsilon},g'_{x_0,\epsilon}) =$$

$$\sum\nolimits_{n,n'\in\mathbf{N}^d}\sum\nolimits_{k,l=1}^{m}\sum\nolimits_{\mu,\nu=1}^{d}\epsilon^{-(|n+n'|)}\int_X dx C_{k,l;\mu,\nu}^{n,n'}(x_0+\epsilon x)\partial_x^n g_{k,\mu}(x)\partial_x^{n'}g'_{l,\nu}(x)$$

$$\forall\ g,g'\in\mathcal{D}_V^m(\Omega). \tag{C.2}$$

where $|n+n'| := \sum_{k=1}^d (n_k+n_k')$: evidently the effective domain of integration here is supp$(g)\cap$supp(g'). Since the functions C are continuous, the summand on the r.h.s. of this equation will diverge, as $\epsilon\rightarrow 0$, unless *either* n and n' are both zero *or* $C_{k,l;\mu,\nu}^{n,n'}(x_0)=0$. Hence the local equilibrium condition (5.21) implies that the only non-vanishing C's are those for which n and n' are zero. Thus, Eq. (C.1) reduces to the form

$$\Gamma(g,g') = \sum_{k,l=1}^m \sum_{\mu,\nu=1}^d \int_\Omega dx C_{k,l;\mu,\nu}^{0,0}(x) g_{k,\mu}(x) g'_{l,\nu}(x) \forall\ g,g'\in\mathcal{D}_V^m(\Omega_1). \tag{C.3}$$

Correspondingly, Eq. (C.2) reduces to the form

$$\Gamma(g_{x_0,\epsilon},g'_{x_0,\epsilon}) = \sum_{k,l=1}^m \sum_{\mu,\nu=1}^d \int_X dx C_{k,l;\mu,\nu}^{0,0}(x_0+\epsilon x) g_{k,\mu}(x) g'_{l,\nu}(x)$$

$$\forall\ g,g'\in\mathcal{D}_V^m(\Omega). \tag{C.4}$$

It now follows immediately from this formula and the local equilibrium condition (5.21) that

$$\sum_{k,l=1}^m \sum_{\mu,\nu=1}^d \int_\Omega dx C_{k,l;\mu,\nu}^{0,0}(x_0) g_{k,\mu}(x) g'_{l,\nu}(x) = 2\big(g, K(\theta(x_0))g'\big)_V$$

$$\forall\ x_0\in\Omega,\ \ g,g'\in\mathcal{D}_V^m(\Omega).$$

Further, in view of Eq. (5.17), this last equation signifies that

$$C_{k,l;\mu,\nu}^{0,0}(x) = 2K_{kl}\big(\theta(x)\big)\delta_{\mu\nu} \tag{C.5}$$

and consequently that Eq. (C.3) reduces to the required formula (5.22), at least for $g,g'\in\mathcal{D}_V^m(\Omega_1)$. The extension to all g,g' in $\mathcal{D}_V^m(\Omega)$ is trivial, since for any pair of elements of the latter space, one can always choose Ω_1 to be an open subset of that space that contains their supports.

Proof of Lemma C.1. Since the test functions $g_{k,\mu}$ and $g'_{l,\nu}$ in Eq. (C.1) are arbitrary elements of $\mathcal{D}(\Omega)$, this lemma reduces to the following one.

Lemma C.2. *Let $\mathcal{T}$ be a $\mathcal{D}'(\Omega^2)$-class distribution whose support lies in the region $\{(x,x')\in\Omega^2|x'=x\}$ and let Ω_1 be an open subset of Ω whose boundary, $\partial\Omega_1$, does not intersect $\partial\Omega$. Then the restriction of $\mathcal{T}$ to the domain $\{f\otimes f'|f,f'\in\mathcal{D}(\Omega_1)\}$ is given by a finite sum of the form*

$$\mathcal{T}(f\otimes f') = \sum_{n,n'\in\mathbf{N}^d} \int_\Omega dx C^{n,n'}(x)\partial_x^n f(x)\partial_x^{n'} f'(x)\ \forall\ f,f'\in\mathcal{D}(\Omega_1), \tag{C.6}$$

where the C's are continuous functions on Ω with supports in some neighbourhood of Ω_1.

Proof of Lemma C.2. Let σ be a $\mathcal{D}(\Omega)$- class function which takes the value unity in Ω_1 and whose support lies in a compact connected subset, K, of Ω whose boundary, ∂K, does not intersect either $\partial\Omega$ or $\partial\Omega_1$. We define the distribution $\tilde{\mathcal{T}}$ $\bigl(\in\mathcal{D}'(\Omega^2)\bigr)$ by the formula

$$\tilde{\mathcal{T}}(x,x')=\sigma(x)\sigma(x')\mathcal{T}(x,x'). \qquad (C.7)$$

Thus, $\tilde{\mathcal{T}}$ coincides with $\mathcal{T}$ in Ω_1^2 and

$$\mathrm{supp}(\tilde{\mathcal{T}})\subset\{(x,x')\in K^2|x'=x\}. \qquad (C.8)$$

We define Φ to be the linear transformation of X^2 given by the formula

$$\Phi(y,z)=(y+z,y-z)\ \forall\ y,z\in X \qquad (C.9)$$

from which its follows that

$$\Phi^{-1}(x,x')=\bigl(\frac{1}{2}(x+x'),\frac{1}{2}(x-x')\bigr)\ \forall\ x,x'\in X. \qquad (C.10)$$

We then define

$$\Theta:=\Phi^{-1}(\Omega^2)=\{(y,z)\in X^2|(y\pm z)\in\Omega\},$$

and we define the bijection $F\rightarrow\hat{F}$ of $\mathcal{D}(\Omega^2)$ onto $\mathcal{D}(\Theta)$ by the formula $\hat{F}=F\circ\Phi$, i.e.

$$\hat{F}(y,z)=F(y+z,y-z)\ \forall\ (y,z)\in\Theta. \qquad (C.11)$$

Correspondingly we define the distribution $\hat{\mathcal{T}}$ $(\in\mathcal{D}'(\Theta))$ in terms of $\tilde{\mathcal{T}}$ by the formula

$$\hat{\mathcal{T}}(\hat{F})=\tilde{\mathcal{T}}(F)\ \forall\ F\in\mathcal{D}(\Omega^2). \qquad (C.12)$$

It follows from Eqs. (C.8), (C.11) and (C.12) that

$$\mathrm{supp}(\hat{\mathcal{T}})\subset K\times\{0\}. \qquad (C.13)$$

We want to restrict $\hat{\mathcal{T}}$ to an open subset of Θ which contains the support of this distribution and takes the form $\Omega_2\times J$, where Ω_2 and J are open subsets of Ω and X respectively. Accordingly, we choose b to be a positive number that is less than $dist(\partial K,\partial\Omega)$, the minimal distance between the boundaries, ∂K and $\partial\Omega$, of K and Ω. We then define $\Omega_2:=\{y\in X|(y,z)\in\Theta\ \forall\ |z|\leq b\}$ and $J:=\{z\in X||z|<b\}$. It follows from these definitions that Ω_2 and $\Omega_2\times J$ are open subsets of Ω and Θ, respectively, that $K\subset\Omega_2$ and that $\partial\Omega_2$, the boundary of Ω_2, does not intersect either ∂K or $\partial\Omega$. Hence, by Eq. (C.13), $\Omega_2\times J$ is an open neighbourhood of $\mathrm{supp}(\hat{\mathcal{T}})$ and the restriction, $\hat{\mathcal{T}}'$, of $\hat{\mathcal{T}}$ to this domain carries all the information we require. It follows from its definition that $\hat{\mathcal{T}}'\in\mathcal{D}'(\Omega_2\times J)$.

Now let e be an arbitrary element of $\mathcal{D}(\Omega_2)$. Then for $e'\in\mathcal{D}(J)$, $\hat{\mathcal{T}}'$ induces a continuous linear functional $\hat{\mathcal{T}}_e'$ on $\mathcal{D}(J)$ according to the formula

$$\hat{\mathcal{T}}_e'(e')=\hat{\mathcal{T}}'(e\otimes e')\ \forall\ e'\in\mathcal{D}(J), \qquad (C.14).$$

where the mapping $e \to \hat{\mathcal{T}}_e'$ of $\mathcal{D}(\Omega_2)$ into $\mathcal{D}'(J)$ is continuous. Further, it follows from Eqs. (C.13) and (C.14) that $\hat{\mathcal{T}}_e'$ has support at the origin and consequently, by Schwartz's point support theorem [33, Theorem 35], that this distribution is a finite sum of derivatives of $\delta(z)$, with coefficients given by linear continuous functionals of e, i.e.

$$\hat{\mathcal{T}}_e'(e') = \sum\nolimits_n T_n(e)(\partial^n e')(0), \tag{C.15}$$

where each $T_n \in \mathcal{D}'(J)$. Further, in view of the definition of $\hat{\mathcal{T}}_e'$, it follows from Eqs. (C.13) and (C.14) that T_n has support in the compact K and therefore, by Schwartz's compact support theorem [33, Theorem 26], it is a finite sum of derivatives of continuous functions on Ω_2 with support in an arbitrary neighbourhood of K. Consequently, by Eq. (C.15), the action of $\hat{\mathcal{T}}'$ on $\mathcal{D}(\Omega \times J)$ is given by a finite sum of the form

$$\hat{\mathcal{T}}'(\hat{F}) = \sum\nolimits_{n',n} \int_{\Omega_2} dy \hat{D}^{n',n}(y) \partial_y^{n'} \partial_z^n \hat{F}(y,z)_{z=0} \ \forall\ \hat{F} \in \mathcal{D}(\Omega_2 \times J), \tag{C.16}$$

where the $\hat{D}'$s are continuous functions on Ω_2 with support in a neighbourhood of K. Hence, as $\hat{\mathcal{T}}'$ is just the restriction of $\hat{\mathcal{T}}$ to $\mathcal{D}(\Omega_2 \times J)$ and since $\tilde{\mathcal{T}}$ coincides with $\mathcal{T}$ in Ω_1^2, it follows from Eqs. (C.9)-(C.12) that Eq. (C.16) is equivalent to the formula

$$\mathcal{T}(F) = \sum\nolimits_{n,n' \in \mathbf{N}^d} \int_\Omega dx C^{n,n'}(x) \partial_x^n \partial_{x'}^{n'} F(x,x')_{|x'=x} \ \forall\ f, f' \in \mathcal{D}(\Omega_1), \tag{C.17}$$

which in turn is equivalent to the required Eq. (C.6).

Appendix D: Proof of Proposition 5.5

Part (a). The characteristic functional for the process ξ is

$$C(f^{(1)},..,f^{(r)};t_1,..,t_r) = E\big[\exp\big(i\sum\nolimits_{k=1}^r \xi_{t_k}(f^{(k)})\big)\big]$$

$$\forall\ f^{(1)},..,f^{(r)} \in \mathcal{D}^m(\Omega);\ t_1,..,t_r \in \mathbf{R},\ r \in \mathbf{N}. \tag{D.1}$$

Equivalently, since the process is stationary,

$$C(f^{(1)},..,f^{(r)};t_1,..,t_r) = E\big[\exp\big(i\sum\nolimits_{k=1}^r \xi_{t_k+t_0}(f^{(k)})\big)\big]$$

$$\forall\ f^{(1)},..,f^{(r)} \in \mathcal{D}^m(\Omega);\ t_0,t_1,..,t_r \in \mathbf{R},\ r \in \mathbf{N}. \tag{D.2}$$

Here we are at liberty to choose t_0 to be any real number and, for any specified set of times $t_1,..,t_r$, we choose it so that $t_1+t_0,.,t_r+t_0$ are all positive. It then follows from Eq. (5.28) that

$$\xi_{t_k+t_0}(f^{(k)}) = \xi(T^\star_{t_k+t_0} f^{(k)}) + \int_0^{t_k+t_0} dw_{u,0}(T^\star_{t_k+t_0-u} f^{(k)})$$

and therefore that Eq. (D.2) may be re-expressed as

$$C(f^{(1)},..,f^{(r)};t_1,..,t_r)=$$

$$E\big[\exp\big(i\sum_{k=1}^{r}\xi(T^{\star}_{t_k+t_0}f^{(k)})\big)\exp\big(i\sum_{k=1}^{r}\int_0^{t_k+t_0}dw_{u,0}(T^{\star}_{t_k+t_0-u}f^{(k)})\big)\big]. \tag{D.3}$$

We now define

$$\tilde{C}(f^{(1)},..,f^{(r)};t_0,t_1,..,t_r)=\exp\big(i\sum_{k=1}^{r}\int_0^{t_k+t_0}dw_{u,0}(T^{\star}_{t_k+t_0-u}f^{(k)})\big) \tag{D.4}$$

and, using the Schwartz inequality, we infer from the last two equations that

$$|C(f^{(1)},..,f^{(r)};t_1,..,t_r)-\tilde{C}(f^{(1)},..,f^{(r)};t_0,t_1,..,t_r)|^2$$

$$\le E\big[|\exp\big(i\sum_{k=1}^{r}\xi(T^{\star}_{t_k+t_0}f^{(k)})\big)-1|^2\big]$$

$$\le E\big[\big(\sum_{k=1}^{r}\xi(T^{\star}_{t_k+t_0}f^{(k)})\big)^2\big]=\sum_{k,l=1}^{r}E\big(\xi(T^{\star}_{t_k+t_0}f^{(k)})\xi(T^{\star}_{t_l+t_0}f^{(l)})\big).$$

It follows from the dissipativity condition (3.12) that the r.h.s. of this estimate vanishes in the limit $t_0\to\infty$, and therefore that

$$C(f^{(1)},..,f^{(r)};t_1,..,t_r)=\lim_{t_0\to\infty}\tilde{C}(f^{(1)},..,f^{(r)};t_0,t_1,..,t_r),$$

i.e., by Eq. (D.4), that

$$C(f^{(1)},..,f^{(r)};t_1,..,t_r)=\lim_{t_0\to\infty}E\big[\exp\big(i\sum_{k=1}^{r}\int_0^{t_k+t_0}dw_{u,0}(T^{\star}_{t_k+t_0-u}f^{(k)})\big)\big].$$

Since, by Eq. (5.7) and the chaoticity condition **(C.1)**, the process w is Gaussian, it follows immediately from this last equation that the process ξ is Gaussian.

In order to show that it is also Markovian, we need just to prove that, for $t\in\mathbf{R}$ and any random variable $B_{\ge t}$ generated by $\{\xi_u(f)|f\in\mathcal{D}^m(\Omega),\ u\ge t\}$, the conditional expectations of $B_{\ge t}$ with respect to the random variables for time t and for times $\le t$ are equal, i.e. that

$$E(B_{\ge t}|\xi_t)=E(B_{\ge t}|\xi_{\le t}). \tag{D.5}$$

Now the random variables over the times $\ge,=$ and $\le t$ are generated by linear combinations of terms

$$F_{\ge t}=\exp\big(i\sum_{k=1}^{p}\xi_{u_k}(f^{(k)})\big), \tag{D.6}$$

$$F_t=\exp\big(i\xi_t(f)\big) \tag{D.7}$$

and

$$F_{\le t}=\exp\big(i\sum_{l=1}^{r}\xi_{s_l}(f'^{(l)})\big), \tag{D.8}$$

respectively, where $u_k \geq t \geq s_l$ and $f^{(k)}$, f and $f'^{(l)}$ are elements of $\mathcal{D}^m(\Omega)$.

It follows from Eqs. (D.1) and (D.6)-(D.8), together with the Gaussian property of ξ, that

$$E(F_{\geq t}F_t) = C(f^{(1)},..,f^{(p)};u_1,..,u_p)C(f;t)\exp\big[-\sum_{k=1}^{p} E\big(\xi_{u_k}(f^{(k)})\xi_t(f)\big)\big] \qquad (D.9)$$

and that

$$E(F_{\geq t}F_{\leq t}) = C(f^{(1)},..,f^{(p)};u_1,..,u_p)C(f'^{(1)},..,f'^{(r)};s_1,..,s_r)\times$$

$$\exp\big[-\sum_{k=1}^{p}\sum_{l=1}^{r} E\big(\xi_{u_k}(f^{(k)})\xi_{s_l}(f'^{(l)})\big)\big]. \qquad (D.10)$$

Further, since $u_k \geq t \geq s_l$, it follows from Eqs. (4.5) and (4.6) that the summands appearing in the exponents in Eqs. (D.9) and (D.10) are equal to $E\big(\xi(T^{\star}_{u_k-t}f^{(k)})\xi(f)\big)$ and $E\big(\xi(T^{\star}_{u_k-s_l}f^{(k)})\xi(f'^{(l)})\big)$, respectively, and therefore those equations may be re-expressed as

$$E(F_{\geq t}F_t) = C(f^{(1)},..,f^{(p)};u_1,..,u_p)C(f;t)\exp\big[-\sum_{k=1}^{p} E\big(\xi(T^{\star}_{u_k-t}f^{(k)})\xi(f)\big)\big] \quad (D.11)$$

and

$$E(F_{\geq t}F_{\leq t}) = C(f^{(1)},..,f^{(p)};u_1,..,u_p)C(f'^{(1)},..,f';s_1,..,s_r)\times$$

$$\exp\big[-\sum_{k=1}^{p}\sum_{l=1}^{r} E\big(\xi(T^{\star}_{u_k-s_l}f^{(k)})\xi(f'^{(l)})\big)\big]. \qquad (D.12)$$

Further, since $E(F_{\geq t}|\xi_t)$ is the unique random variable of the ξ-process at time t for which

$$E\big(E(F_{\geq t}|\xi_t)F_t\big) = E(F_{\geq t}F_t)$$

for all $F_{\geq t}$ and F_t of the forms given by Eqs. (D.6) and (D.7), respectively, it follows from Eq. (D.9), together with the stationarity and the Gaussian property of the process, that

$$E(F_{\geq t}|\xi_t) = \frac{C(f^{(1)},..,f^{(p)};u_1,..,u_p)}{C(T^{\star}_{u_1-t}f^{(1)},..,T^{\star}_{u_p-t}f^{(p)};0,..,0)}\exp\big(i\sum_{k=1}^{p}\xi_t(T^{\star}_{u_k-t}f^{(k)}\big). \qquad (D.13)$$

Hence, by Eq. (D.8),

$$E\big(E(F_{\geq t}|\xi_t)F_{\leq t}\big) =$$

$$\frac{C(f^{(1)},..,f^{(p)};u_1,..,u_p)}{C(T^{\star}_{u_1-t}f^{(1)},..,T^{\star}_{u_p-t}f^{(p)};0,..,0)}C\big(f'^{(1)},..,f'^{(r)},\sum_{k=1}^{p}T^{\star}_{u_k-t}f^{(k)}:s_1,..,s_r,t\big). \quad (D.14)$$

Further, in view of the Gaussian property of the process, the last factor in this formula is equal to

$$C(f'^{(1)},..,f'^{(r)}:s_1,..,s_r)C(\sum_{k=1}^{p}T^{\star}_{u_k-t}f^{(k)};t)\times$$

$$\exp\big[-\sum_{k=1}^{p}\sum_{l=1}^{r} E\big(\xi_t(T^{\star}_{u_k-t}f^{(k)}\xi_{s_l}(f'^{(l)})\big)\big].$$

Therefore since, by Eq. (4.6) and the semigroup property of $T^\star(\mathbf{R}_+)$, the summand in the exponent in this expression is equal to $E\big(\xi(T^\star_{u_k-s_l}f^{(k)}\xi(f'^{(l)})\big)$, it follows from Eqs. (D.10) and (D.14) that

$$E\big(E(F_{\geq t}|\xi_t)F_{\leq t}\big) = E(F_{\geq t}F_{\leq t}).$$

Hence

$$E(F_{\geq t}|\xi_t) = E(F_{\geq t}|\xi_{\leq t}),$$

which signifies that the process is temporally Markovian.

Part (b). Since Eq. (5.8) implies that $w_{t,s} = -w_{s,t}$ and since $w_{t,s}$ and ξ_u are Gaussian random fields whose means are zero, it follows from Eq. (5.9) that the latter two fields are statistically independent of one another if s and t are both greater than or equal to u.

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